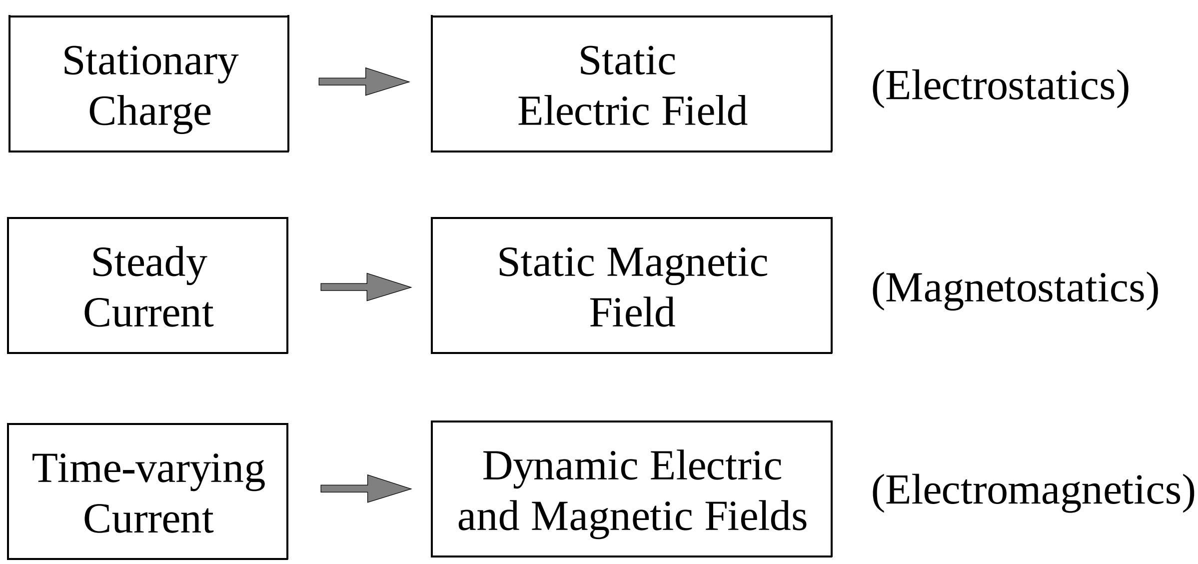
Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Magnetostatic Fields

According to *Coulomb's law*, any distribution of stationary charge produces a static electric field (electrostatic field). The analogous equation to Coulomb's law for electric fields is the *Biot-Savart law* for magnetic fields. The Biot-Savart law shows that when charge moves at a constant rate (direct current - DC), a static magnetic field (magnetostatic field) is produced. When the rate of charge movement varies with time (for example, an alternating current - AC), we find that coupled electric and magnetic fields are produced (electromagnetic field).



Static magnetic fields are also produced by stationary permanent magnets. When permanent magnets are set in motion such that a time-varying magnetic field is produced, a time-varying electric field is simultaneously produced. A time-varying electric field cannot exist without a corresponding time-varying magnetic field and vice versa.

All of the previously defined equations related to the electric field have *dual* equations related to the magnetic field. All of the magnetic field terms in these dual equations have dual units to those electric field terms in the electric field equations.

The so-called *constitutive equations* that relate the electric field to the electric flux density and the magnetic field to the magnetic flux density are

$$\boldsymbol{D} = \epsilon \boldsymbol{E} = \epsilon_r \epsilon_o \boldsymbol{E}$$

$\boldsymbol{E}$ - vector electric field (V/m)

$\boldsymbol{D}$ - vector electric flux density (C/m^2)

ϵ - total permittivity (F/m)

ϵ_r - relative permittivity (unitless)

$\epsilon_o = 8.854 \times 10^{-12}$ F/m - free space permittivity

$$\boldsymbol{B} = \mu \boldsymbol{H} = \mu_r \mu_o \boldsymbol{H}$$

$\boldsymbol{H}$ - vector magnetic field (A/m)

$\boldsymbol{B}$ - vector magnetic flux density (T = Wb/m^2)

μ - total permeability (H/m)

μ_r - relative permeability (unitless)

$\mu_o = 4\pi \times 10^{-7}$ H/m - free space permeability

T $\equiv$ Tesla

Wb $\equiv$ Weber

The third constitutive equation is the relationship between current density and electric field.

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

The three medium characteristics (μ, ϵ, σ) are known as the *constitutive parameters*.

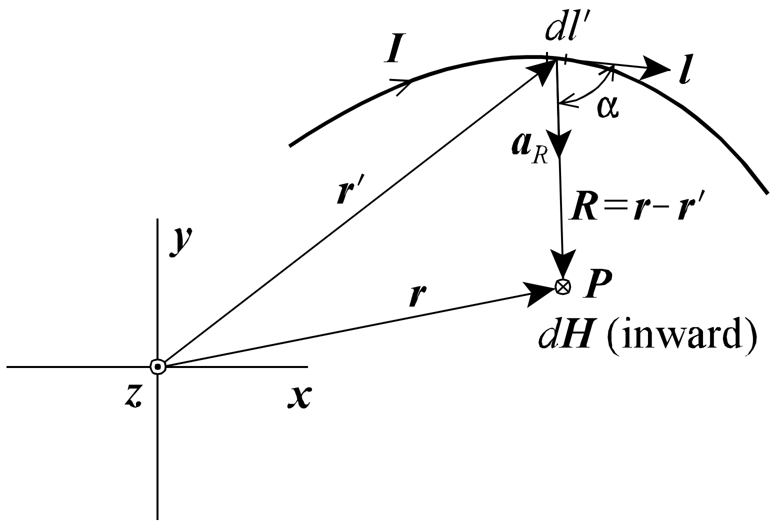
Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Biot-Savart Law

The Biot-Savart law defines the magnetostatic field produced by a steady current. The overall magnetic field produced by an arbitrary vector line current $\boldsymbol{I}$ (filament, zero cross-section) is expressed as a line integral of the current. According to the Biot-Savart law, the differential vector magnetic field ($d\boldsymbol{H}$) at the field point P produced by a differential element of current $\boldsymbol{I}dl'$ is

$$d\boldsymbol{H} = \frac{\boldsymbol{I}\times\boldsymbol{a}_R}{4\pi R^2}\,dl'$$

$$= \frac{\boldsymbol{I}\times\boldsymbol{R}}{4\pi R^3}\,dl'$$



or

$$dH = \frac{I\sin\alpha}{4\pi R^2}\,dl'$$

where

$\boldsymbol{r}$ - vector locating the field point P

$\boldsymbol{r}'$ - vector locating the source point ($\boldsymbol{I}dl'$)

$\boldsymbol{R} = \boldsymbol{r}-\boldsymbol{r}'$ (vector from the source point to the field point)

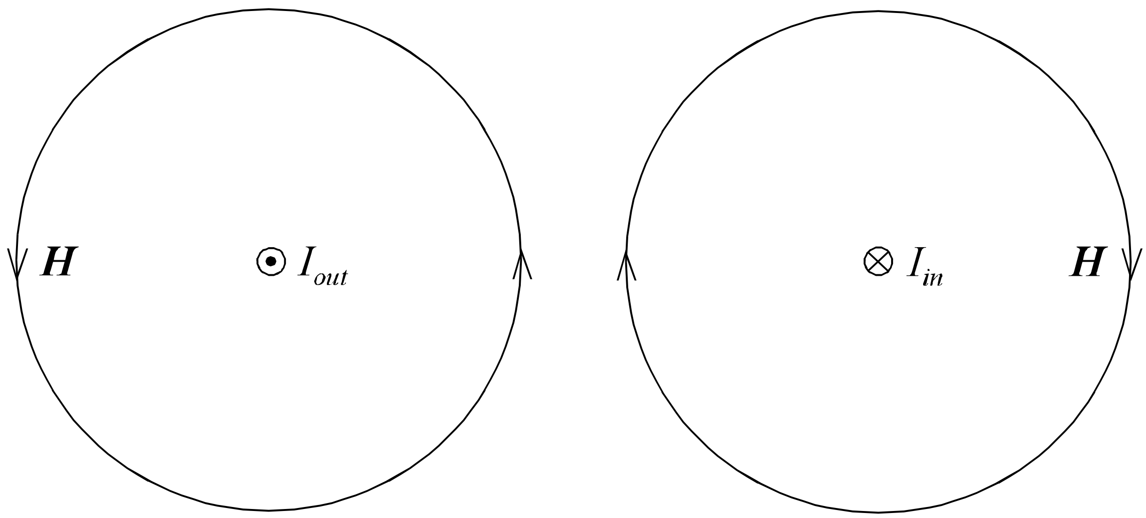
$R = |\boldsymbol{R}| = |\boldsymbol{r}-\boldsymbol{r}'|$ (distance from the source point to the field point)

$\boldsymbol{a}_R = \boldsymbol{R}/R$ (unit vector in the direction of $\boldsymbol{R}$)

$\boldsymbol{l}$ - unit vector in the direction of the current $\boldsymbol{I}$ at dl'

α - angle between $\boldsymbol{l}$ and $\boldsymbol{a}_R$

Note that the direction of the magnetic field is given by the direction of $\boldsymbol{I}\times\boldsymbol{R}$. For infinite length line currents, the magnetic field direction is given by the right hand rule.



The scalar and vector forms of the total magnetic field for an arbitrary line current $\boldsymbol{I}$ (A) are given by

Scalar

$$H = \frac{I}{4\pi}\int_L \frac{\sin\alpha}{R^2}\,dl'$$

Vector

$$\boldsymbol{H} = \frac{1}{4\pi}\int_L \frac{\boldsymbol{I}\times\boldsymbol{a}_R}{R^2}\,dl' = \frac{1}{4\pi}\int_L \frac{\boldsymbol{I}\times\boldsymbol{R}}{R^3}\,dl'$$

The scalar and vector forms of the total magnetic field for an arbitrary surface current ***K*** (A/m) are given by

Scalar

$$H = \frac{1}{4\pi} \iint_S \frac{K \sin\alpha}{R^2}\, ds'$$

Vector

$$\boldsymbol{H} = \frac{1}{4\pi} \iint_S \frac{\boldsymbol{K} \times \boldsymbol{a}_R}{R^2}\, ds' = \frac{1}{4\pi} \iint_S \frac{\boldsymbol{K} \times \boldsymbol{R}}{R^3}\, ds'$$

The scalar and vector forms of the total magnetic field for an arbitrary volume current ***J*** (A/m^2) are given by

Scalar

$$H = \frac{1}{4\pi} \iiint_V \frac{J \sin\alpha}{R^2}\, dv'$$

Vector

$$\boldsymbol{H} = \frac{1}{4\pi} \iiint_V \frac{\boldsymbol{J} \times \boldsymbol{a}_R}{R^2}\, dv' = \frac{1}{4\pi} \iiint_V \frac{\boldsymbol{J} \times \boldsymbol{R}}{R^3}\, dv'$$

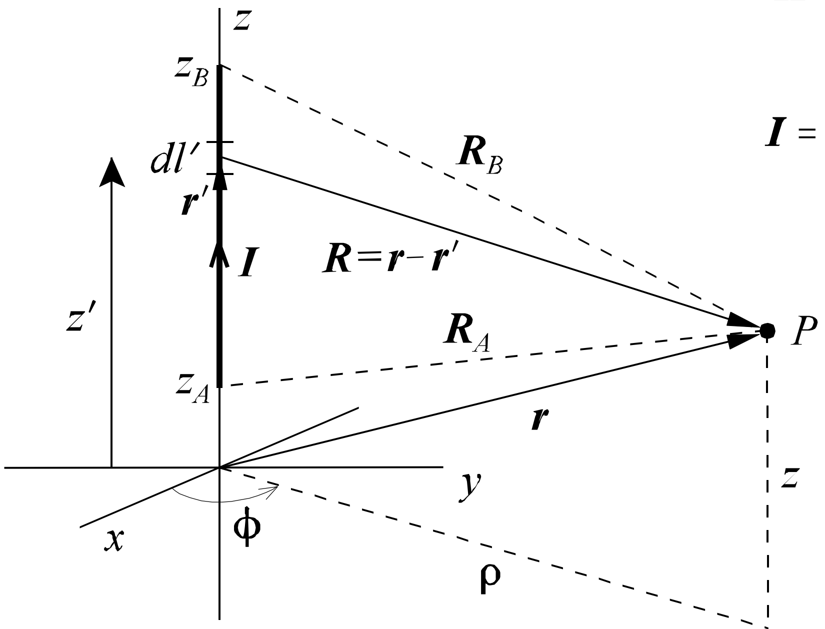
Note that the magnetic field is proportional to the product of the current and the differential element in each case. This product is defined as the current moment and has units of A-m.

$$I dl' \quad \Leftrightarrow \quad K ds' \quad \Leftrightarrow \quad J dv' \qquad \textbf{(current moment)}$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

<u>Example</u> (Biot-Savart law / line current)

Determine the magnetic field of a line segment of current lying along the z-axis extending from $z = z_A$ to $z = z_B$.



$$\boldsymbol{H} = \frac{1}{4\pi}\int_L \frac{\boldsymbol{I}\times\boldsymbol{R}}{R^3}\,dl'$$

$$\boldsymbol{I} = I\boldsymbol{a}_z \qquad dl' = dz'$$

$$\boldsymbol{r} = \rho\,\boldsymbol{a}_\rho + z\,\boldsymbol{a}_z$$

$$\boldsymbol{r}' = z'\boldsymbol{a}_z$$

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}' = \rho\,\boldsymbol{a}_\rho + (z - z')\boldsymbol{a}_z$$

$$R = |\boldsymbol{r} - \boldsymbol{r}'| = \sqrt{\rho^2 + (z - z')^2}$$

$$\boldsymbol{H} = \frac{1}{4\pi}\int_{z_A}^{z_B} \frac{(I\boldsymbol{a}_z)\times[\rho\,\boldsymbol{a}_\rho + (z-z')\boldsymbol{a}_z]}{[\rho^2 + (z-z')^2]^{3/2}}\,dz'$$

$$= \frac{1}{4\pi}\int_{z_A}^{z_B} \frac{I\rho\,\boldsymbol{a}_\phi}{[\rho^2 + (z-z')^2]^{3/2}}\,dz'$$

$$= \frac{I\rho\,\boldsymbol{a}_\phi}{4\pi}\int_{z_A}^{z_B} \frac{dz'}{[\rho^2 + (z-z')^2]^{3/2}}$$

(Given the field point P, the direction direction of $\boldsymbol{a}_\phi$ does not change as z is varied along the length of the current)

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Transformation of variable:

$$\text{let } \alpha = z - z^{\prime} \qquad d\alpha = -dz^{\prime}$$

$$z^{\prime} = z_A \quad \Rightarrow \quad \alpha = z - z_A$$

$$z^{\prime} = z_B \quad \Rightarrow \quad \alpha = z - z_B$$

$$\boldsymbol{H} = \frac{I\rho\, \boldsymbol{a}_\phi}{4\pi} \int_{z-z_A}^{z-z_B} \frac{-d\alpha}{(\alpha^2+\rho^2)^{3/2}} = -\frac{I\rho\, \boldsymbol{a}_\phi}{4\pi} \left[\frac{\alpha}{\rho^2(\alpha^2+\rho^2)^{1/2}} \right]_{z-z_A}^{z-z_B}$$

$$= -\frac{I}{4\pi\rho} \left[\frac{z-z_B}{[\rho^2+(z-z_B)^2]^{1/2}} - \frac{z-z_A}{[\rho^2+(z-z_A)^2]^{1/2}} \right] \boldsymbol{a}_\phi$$

$$= \frac{I}{4\pi\rho} \left[\frac{z-z_A}{R_A} - \frac{z-z_B}{R_B} \right] \boldsymbol{a}_\phi$$

Given this general result for the magnetic field of a current segment, we may apply it to several special cases.

(1) Line current, symmetric about the *x*-*y* plane, $z_A = -z_o$, $z_B = z_o$

$$\boldsymbol{H} = \frac{I}{4\pi\rho} \left[\frac{z+z_o}{R_A} - \frac{z-z_o}{R_B} \right] \boldsymbol{a}_\phi$$

If the field point lies in the *x*-*y* plane, then
$z = 0$, $R_A = R_B = R_o = [\rho^2 + z_o^2]^{1/2}$

$$\boldsymbol{H} = \frac{I}{4\pi\rho} \left[\frac{2z_o}{R_o} \right] = \frac{I z_o}{2\pi\rho\sqrt{\rho^2+z_o^2}}\, \boldsymbol{a}_\phi$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

(2) Semi-infinite line current, $z' \in (0, \infty)$
$z_A = 0$, $z_B = \infty$, $R_A = [\rho^2 + z^2]^{1/2}$, $R_B = \infty$

$$\lim_{z_B \to \infty} \frac{z - z_B}{R_B} = \lim_{z_B \to \infty} \frac{z - z_B}{[\rho^2 + (z - z_B)^2]^{1/2}}$$

$$= \lim_{z_B \to \infty} \frac{\dfrac{z}{z_B} - 1}{\left[\dfrac{\rho^2}{z_B^2} + \dfrac{(z - z_B)^2}{z_B^2}\right]^{1/2}} = -1$$

$$\boldsymbol{H} = \frac{I}{4\pi\rho}\left[1 + \frac{z}{\sqrt{\rho^2 + z^2}}\right]\boldsymbol{a}_\phi$$

If the field point lies in the x-y plane ($z = 0$),

$$\boldsymbol{H} = \frac{I}{4\pi\rho}\boldsymbol{a}_\phi$$

(3) Infinite line current, $z' \in (-\infty, \infty)$
Wherever the field point is located, the infinite length line current can be viewed as two semi-infinite line currents. The resulting magnetic field is twice that of the semi-infinite length current segment.

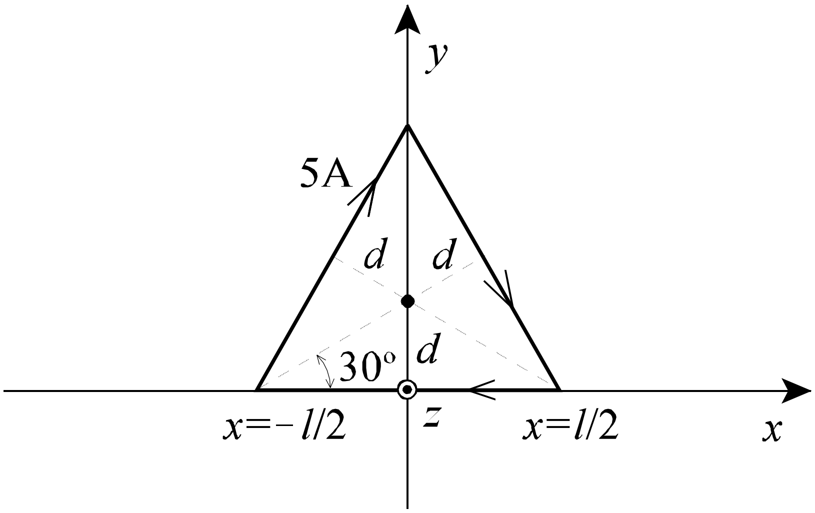
$$\boldsymbol{H} = 2\frac{I}{4\pi\rho}\boldsymbol{a}_\phi = \frac{I}{2\pi\rho}\boldsymbol{a}_\phi$$

The previous formulas are useful when determining the magnetic field of a closed current loop made up of straight segments. The principle of superposition may be applied to determine the total magnetic field produced by the loop. The total magnetic field produced by the loop is the vector sum of the magnetic field contributions from each current segment.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed-ucoz.org

Example (Current loop / straight segments)

Determine the magnetic field at the center of the current loop in the shape of an equilateral triangle (side length l = 4m) carrying a steady current of 5A.



$$\tan 30^o = \frac{d}{l/2}$$

$$d = (l/2)\tan 30^o = 2\tan 30^o = 1.155\,\text{m}$$

$$\boldsymbol{H}_{segment} = \frac{I z_o}{2\pi\rho\sqrt{\rho^2 + z_o^2}}(-\boldsymbol{a}_z) \qquad z_o = l/2 = 2\,\text{m} \qquad \rho = d = 1.155\,\text{m}$$

$$\boldsymbol{H}_{total} = 3\boldsymbol{H}_{segment} = 3\,\frac{(5)(2)}{2\pi(1.155)\sqrt{1.155^2 + 2^2}}(-\boldsymbol{a}_z)$$

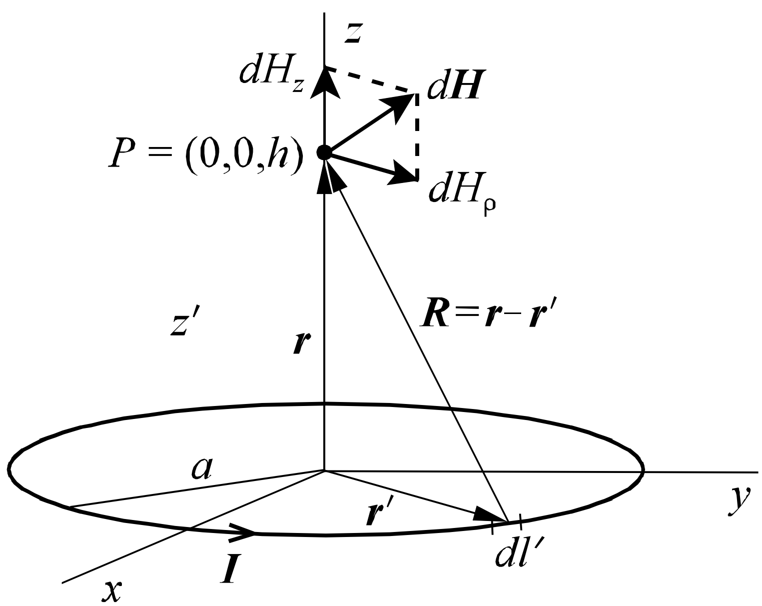
$$= -1.79\,\boldsymbol{a}_z\ (\text{A/m})$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Magnetic Field Due to a Circular Current Loop

The Biot-Savart law can be used to determine the magnetic field at the center of a circular loop of steady current.

$$\boldsymbol{H} = \frac{1}{4\pi}\int_L \frac{\boldsymbol{I}\times\boldsymbol{R}}{R^3}\,dl'$$



$$\boldsymbol{I} = I\boldsymbol{a}_\phi$$

$$dl' = a\,d\phi'$$

$$\boldsymbol{r} = h\boldsymbol{a}_z$$

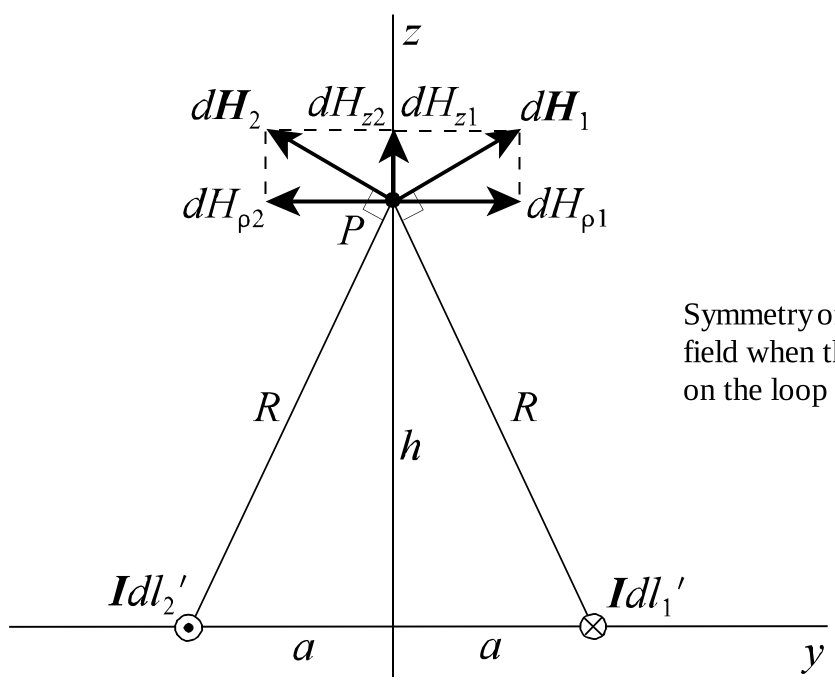
$$\boldsymbol{r}' = a\boldsymbol{a}_\rho$$

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$$

$$= h\boldsymbol{a}_z - a\boldsymbol{a}_\rho$$

$$R = |\boldsymbol{r} - \boldsymbol{r}'|$$

$$= \sqrt{a^2 + h^2}$$



Symmetry of the loop magnetic field when the field point lies on the loop axis

$$\boldsymbol{H} = \frac{1}{4\pi}\int_0^{2\pi} \frac{(I\boldsymbol{a}_\phi)\times(h\boldsymbol{a}_z - a\boldsymbol{a}_\rho)}{(a^2+h^2)^{3/2}}\, a\,d\phi'$$

$$= \frac{I}{4\pi}\int_0^{2\pi} \frac{h\boldsymbol{a}_\rho + a\boldsymbol{a}_z}{(a^2+h^2)^{3/2}}\, a\,d\phi'$$

$$\boldsymbol{a}_\rho = \cos\phi'\,\boldsymbol{a}_x + \sin\phi'\,\boldsymbol{a}_y$$

$$\boldsymbol{H} = \frac{Ia}{4\pi(a^2+h^2)^{3/2}}\left[h\boldsymbol{a}_x \underbrace{\int_0^{2\pi}\cos\phi'\,d\phi'}_{0} + h\boldsymbol{a}_y \underbrace{\int_0^{2\pi}\sin\phi'\,d\phi'}_{0} + a\boldsymbol{a}_z\int_0^{2\pi} d\phi'\right]$$

The integrals for the *x* and *y* components of the magnetic field are zero. These field components can be shown to be zero by symmetry.

$$\boldsymbol{H} = \frac{Ia}{4\pi(a^2+h^2)^{3/2}}\left[2\pi a\boldsymbol{a}_z\right]$$

$$= \frac{Ia^2}{2(a^2+h^2)^{3/2}}\,\boldsymbol{a}_z$$

At the loop center ($h=0$),

$$\boldsymbol{H} = \frac{I}{2a}\,\boldsymbol{a}_z$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Ampere's Law

Gauss's law is the Maxwell equation that relates the electrostatic field (flux) to the source of the electrostatic field (charge). Ampere's law is the Maxwell equation that relates the magnetostatic field (flux) to the source of the magnetostatic field (current).

$$\oiint_S \mathbf{D}\cdot d\mathbf{s} = Q \qquad (\text{Gauss's law - integral form})$$

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = I \qquad (\text{Ampere's law - integral form})$$

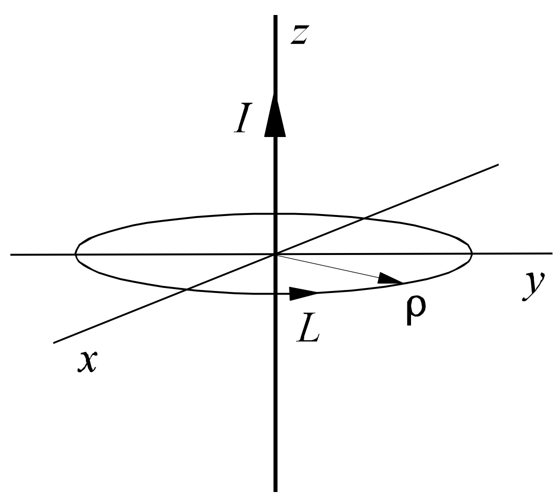
Ampere's Law **-** The line integral of the magnetic field around a closed path equals the net current enclosed (the current direction is implied by the direction of the path according to the right hand rule).

<u>Example</u> (Ampere's law / infinite-length line current)

Given a infinite-length line current I lying along the z-axis, use Ampere's law to determine the magnetic field by integrating the magnetic field around a circular path of radius ρ lying in the x-y plane.

From Ampere's law,

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = \oint_L H_\phi\, dl = I$$



By symmetry, the magnetic field is uniform on the given path so that

$$H_\phi \oint_L dl = H_\phi (2\pi\rho) = I$$

or

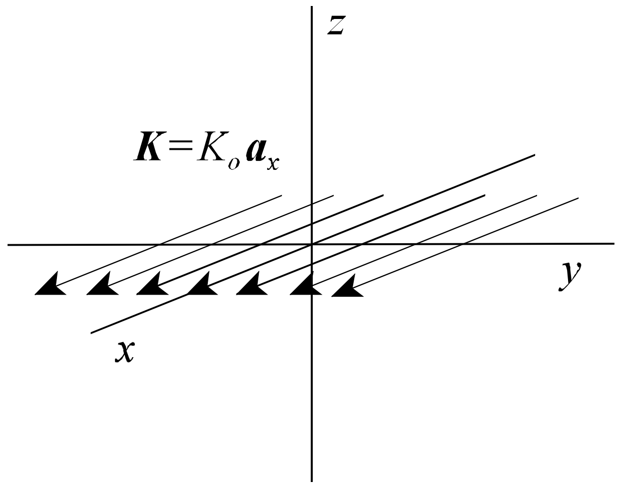
$$H_\phi = \frac{I}{2\pi\rho}$$

This result agrees with that found using the Biot-Savart law.

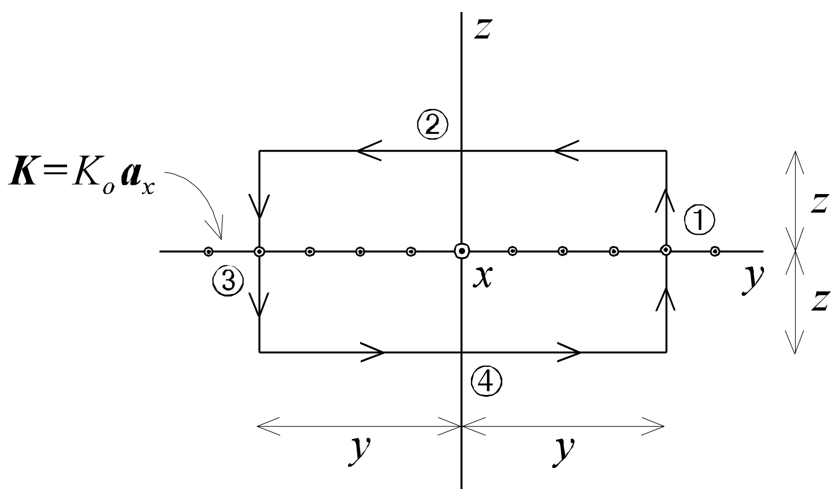
Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed-ucoz.org

Example (Ampere's law / magnetic field of a surface current)

Determine the vector magnetic field produced by a uniform $\boldsymbol{a}_x$-directed surface current covering the x-y plane



Ampere's law may be applied on the path shown below. Note that the path dimensions are defined in terms of arbitrary distances ($-y$,$+y$) and ($-z$,$+z$) such that the result is valid for any value of y and z.

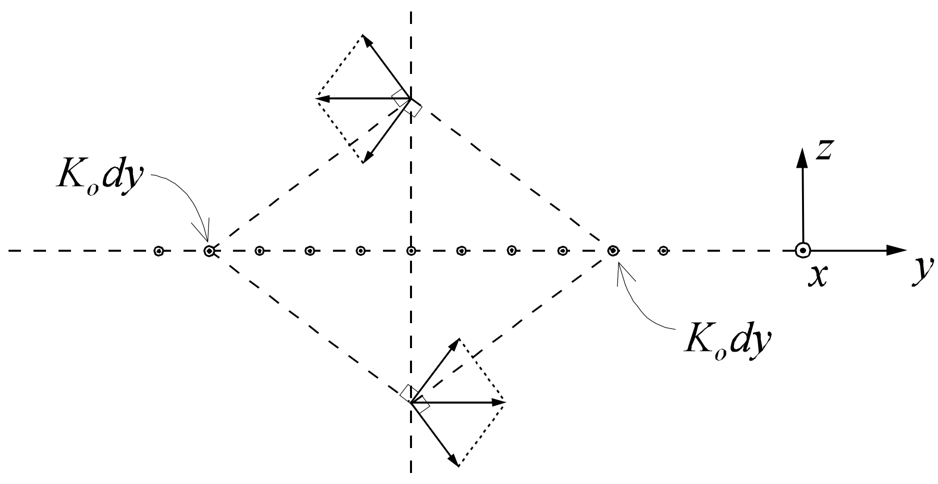


$$\oint_L \boldsymbol{H}\cdot d\boldsymbol{l} = \int_{①} \boldsymbol{H}\cdot d\boldsymbol{l} + \int_{②} \boldsymbol{H}\cdot d\boldsymbol{l} + \int_{③} \boldsymbol{H}\cdot d\boldsymbol{l} + \int_{④} \boldsymbol{H}\cdot d\boldsymbol{l} = I_{enclosed}$$

To evaluate the Ampere's law integral, we must first determine the vector characteristics of the magnetic field. By symmetry, the magnetic field on the horizontal segments of the path (② and ④) must be uniform.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Also by symmetry, we can show that the magnetic field above the surface current is everywhere $-\boldsymbol{a}_y$ directed while the magnetic field everywhere below the surface current is $+\boldsymbol{a}_y$ directed. The overall surface current can be subdivided into differential lengths ($K_o dy$) with each differential length equivalent to a line current. For any given field point, and any given line current, there is always another line current in the opposite direction that produces a magnetic field component that when added to the magnetic field component of the original line current, produces only a $-\boldsymbol{a}_y$ component (above) or $+\boldsymbol{a}_y$ component (below) of magnetic field.



With only horizontal components of magnetic field, the Ampere's law integrals on the vertical paths (① and ③) are zero. The magnetic field on the horizontal paths (② and ④) may be written as

$$\boldsymbol{H} = \begin{cases} -H_o \boldsymbol{a}_y & (z > 0) \\ H_o \boldsymbol{a}_y & (z < 0) \end{cases}$$

where H_o is a constant (the magnetic field is uniform). Given the magnetic field characteristics on the horizontal and vertical paths, the Ampere's law integral can be evaluated to determine the magnetic field of the surface current.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

$$\oint_L \boldsymbol{H}\cdot d\boldsymbol{l} = \int_{②} \boldsymbol{H}\cdot d\boldsymbol{l} + \int_{④} \boldsymbol{H}\cdot d\boldsymbol{l}$$

$$= \int_{+y}^{-y} (-H_o\boldsymbol{a}_y)\cdot(-dy)(-\boldsymbol{a}_y) + \int_{-y}^{+y} (H_o\boldsymbol{a}_y)\cdot(dy)(\boldsymbol{a}_y)$$

$$= -H_o\int_{+y}^{-y} dy + H_o\int_{-y}^{+y} dy$$

$$= -H_o(-2y) + H_o(2y) = H_o(4y)$$

According to Ampere's law, this integral is equal to the total current enclosed by the path. The total current enclosed is the surface current located between $-y$ and $+y$. Given a uniform surface current, the total current is the product of the surface current density and the distance ($2y$) such that

$$I_{enclosed} = K_o(2y)$$

Given the direction of the integration path, the direction of the enclosed current is the $+\boldsymbol{a}_x$ direction. From Ampere's law,

$$\oint_L \boldsymbol{H}\cdot d\boldsymbol{l} = H_o(4y) = I_{enclosed} = K_o(2y) \qquad \Rightarrow \qquad H_o = \frac{K_o}{2}$$

The magnetic field above and below the current sheet is

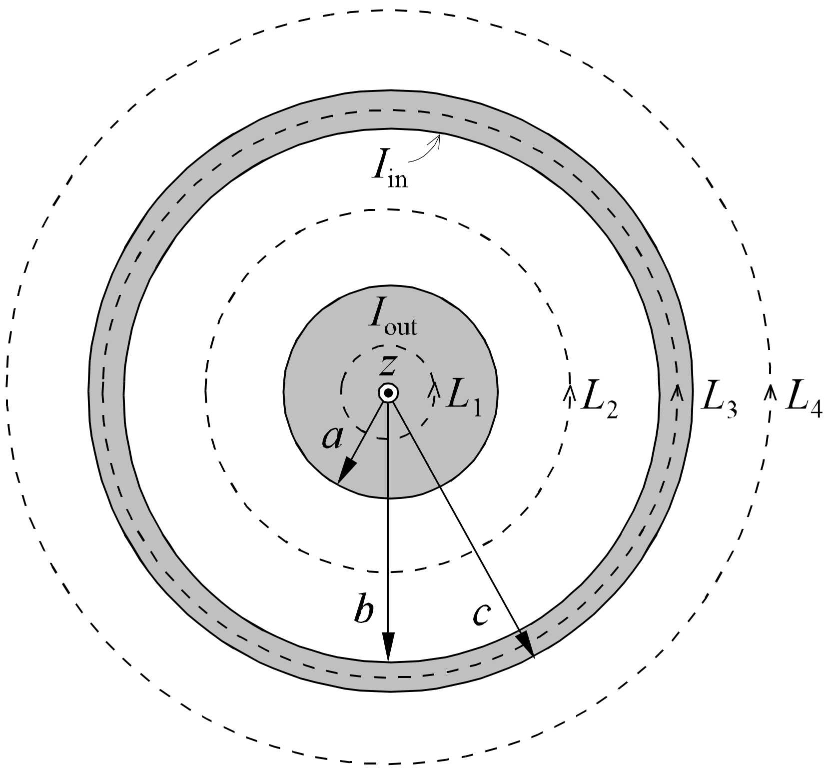
$$\boldsymbol{H} = \begin{cases} -\dfrac{K_o}{2}\boldsymbol{a}_y & (z>0) \\ \dfrac{K_o}{2}\boldsymbol{a}_y & (z<0) \end{cases}$$

Note that the magnetic field is independent of the distance x so that the magnetic field is uniform above and below the surface current.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Example (Ampere's law / Magnetic field in a coaxial transmission line)

The coaxial transmission line shown below carries a total current I in the $+\mathbf{a}_z$ direction through the inner conductor and in the $-\mathbf{a}_z$ direction through the inner conductor. Assume uniform current densities in both conductors of the coaxial transmission line. Use Ampere's law to determine the magnetic field everywhere.



The uniform vector current densities in the inner and outer conductors of the coaxial transmission line ($\mathbf{J}_i$ and $\mathbf{J}_o$, respectively) are

inner conductor $$\mathbf{J}_i = \frac{I}{A_i}\mathbf{a}_z = \frac{I}{\pi a^2}\mathbf{a}_z = J_i\,\mathbf{a}_z$$

outer conductor $$\mathbf{J}_o = \frac{I}{A_o}(-\mathbf{a}_z) = -\frac{I}{\pi(c^2-b^2)}\mathbf{a}_z = -J_o\,\mathbf{a}_z$$

To determine the magnetic field everywhere, Ampere's law is applied on circular paths in the four distinct regions for the coaxial transmission line.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

L_1 $(\rho < a)$ **[region within the inner conductor]**

L_2 $(a < \rho < b)$ **[region between the conductors]**

L_3 $(b < \rho < c)$ **[region within the outer conductor]**

L_4 $(\rho > c)$ **[region outside the outer conductor]**

By symmetry, on all four of the integration paths, the magnetic field is uniform and has only an $\boldsymbol{a}_\phi$ component. Thus, for each path, Ampere's law reduces to

$$\oint_L \boldsymbol{H}\cdot d\boldsymbol{l} = \oint_L H_\phi \, dl = H_\phi \oint_L dl = H_\phi (2\pi\rho) = I_{enclosed}$$

or

$$H_\phi = \frac{I_{enclosed}}{2\pi\rho} \qquad \boldsymbol{H} = \frac{I_{enclosed}}{2\pi\rho}\boldsymbol{a}_\phi$$

The magnetic field in each region is proportional to the net current enclosed by the path.

<u>Within the inner conductor</u> $(\rho < a)$ $[L_1]$

$$I_{enclosed} = \iint_S \boldsymbol{J}\cdot d\boldsymbol{s} = \iint_S (J_i \boldsymbol{a}_z)\cdot(ds\,\boldsymbol{a}_z) = J_i \iint_S ds = J_i(\pi\rho^2)$$

$$= \frac{I}{\pi a^2}(\pi\rho^2) = I\frac{\rho^2}{a^2}$$

$$\boldsymbol{H} = \frac{I_{enclosed}}{2\pi\rho}\boldsymbol{a}_\phi = \frac{I\rho^2}{2\pi\rho a^2}\boldsymbol{a}_\phi = \frac{I\rho}{2\pi a^2}\boldsymbol{a}_\phi \qquad (\rho < a)$$

Between the conductors ($a < \rho < b$) [L_2]

$$I_{enclosed} = I$$

$$\boldsymbol{H} = \frac{I_{enclosed}}{2\pi\rho}\,\boldsymbol{a}_\phi = \frac{I}{2\pi\rho}\,\boldsymbol{a}_\phi \qquad (a < \rho < b)$$

Within the outer conductor ($b < \rho < c$) [L_3]

$$I_{enclosed} = I + \iint_{S_o} \boldsymbol{J}\cdot d\boldsymbol{s} = I + \iint_{S_o} (-J_o\,\boldsymbol{a}_z)\cdot(ds\,\boldsymbol{a}_z) = I - J_o \iint_{S_o} ds$$

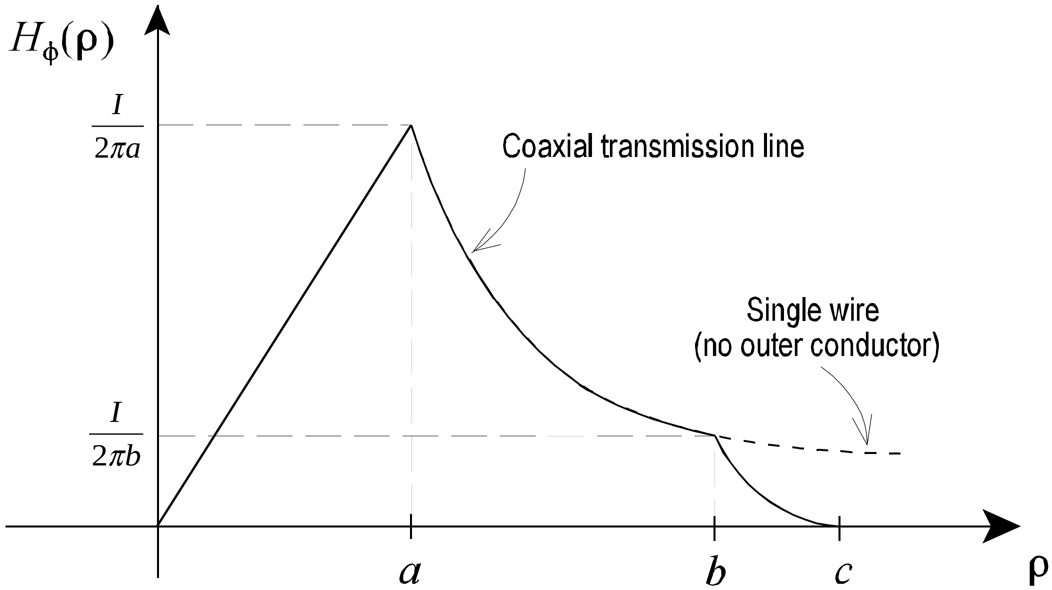
$$= I - \left[\frac{I}{\pi(c^2 - b^2)}\right]\left[\pi(\rho^2 - b^2)\right] = I\left[1 - \frac{\rho^2 - b^2}{c^2 - b^2}\right]$$

$$\boldsymbol{H} = \frac{I_{enclosed}}{2\pi\rho}\,\boldsymbol{a}_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{c^2 - b^2}\right]\boldsymbol{a}_\phi \qquad (b < \rho < c)$$

Outside the outer conductor ($\rho < c$) [L_4]

$$I_{enclosed} = I + (-I) = 0$$

$$\boldsymbol{H} = \frac{I_{enclosed}}{2\pi\rho}\,\boldsymbol{a}_\phi = 0 \qquad (\rho > c)$$



According to the equations obtained from Ampere's law, the magnetic field in the four regions of the coaxial transmission line is

$$\boldsymbol{H} = \frac{I\rho}{2\pi a^2}\,\boldsymbol{a}_\phi \qquad (\rho < a)$$

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\,\boldsymbol{a}_\phi \qquad (a < \rho < b)$$

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{c^2 - b^2}\right]\boldsymbol{a}_\phi \qquad (b < \rho < c)$$

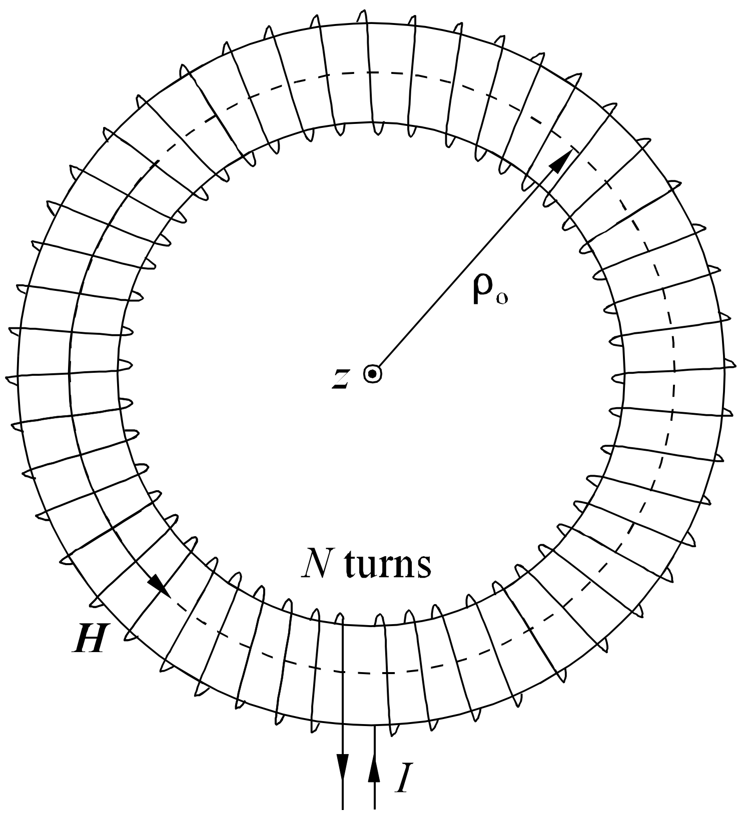
$$\boldsymbol{H} = 0 \qquad (\rho > c)$$

The magnetic field of a single conductor carrying a uniform current density can be determined from the results of the coaxial transmission line. With no outer conductor, the curve for the region between the coaxial conductors would continue for the single conductor.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Toroid

Another commonly encountered magnetic energy storage geometry is the *toroid*. A toroid is formed by wrapping a conductor around a ring of uniform cross-section (typically circular cross-section).



The distance from the center of the ring to the center of the ring cross-section is defined as the *mean radius* ρ_o. Given a circular cross-section of radius a, if the mean radius is large relative to the radius of the cross section ($\rho_o \gg a$), then the toroid may be viewed as a long solenoid bent into the shape of a circle (the magnetic field within the toroid may be assumed to be uniform). Application of Ampere's law on the mean radius path gives

$$\oint_L \boldsymbol{H}\cdot d\boldsymbol{l} = \oint_L H_\phi\, dl = H_\phi \oint_L dl = H_\phi(2\pi\rho_o) = I_{enclosed} = NI$$

Solving for the toroid magnetic field yields

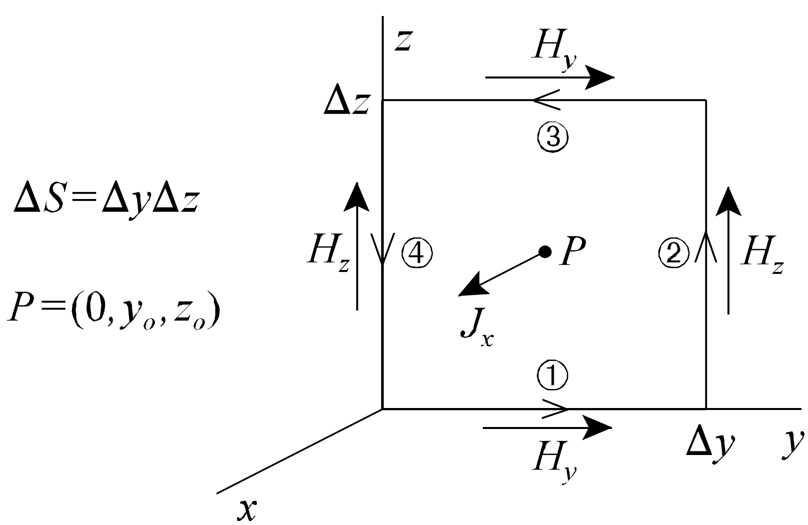
$$H_\phi = \frac{NI}{2\pi\rho_o} = \frac{NI}{l}$$

where $l = 2\pi\rho_o$ is the equivalent length of the toroid. The magnetic field at any point within the toroid is the same as that found at the center of the long solenoid. The primary advantage of the toroid over the solenoid is the confinement of the magnetic field within the toroid as opposed to the solenoid which produces magnetic fields external to the coil. Also, the toroid does not suffer from the end effects (fringing) seen in the solenoid.

Differential Form of Ampere's Law

(Curl Operator)

The differential form of Ampere's law may be determined by applying Ampere's law to a differential surface. Given the differential surface shown below, integration of the magnetic field around the path L gives



$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = I_{enclosed} = J_x \Delta S$$

Dividing the Ampere's law integral by the differential surface ΔS and taking the limit as ΔS approaches zero defines the x-component of the *curl* operator.

$$\boldsymbol{a}_x \cdot \text{curl}\,\boldsymbol{H} \equiv \lim_{\Delta S \to 0} \frac{\oint \boldsymbol{H} \cdot d\boldsymbol{l}}{\Delta S} = J_x$$

To evaluate the Ampere's law integral needed to define the x-component of the curl operator, the magnetic field is first defined at the point P.

$$\boldsymbol{H}(P) = H_{xo}\boldsymbol{a}_x + H_{yo}\boldsymbol{a}_y + H_{zo}\boldsymbol{a}_z$$

Expanding the y and z components of the magnetic field in a Taylor series about the point P gives

$$H_y(z) = H_{yo} + \frac{\partial H_y}{\partial z}(z - z_o) + \frac{\partial^2 H_y}{\partial z^2}\frac{(z - z_o)^2}{2!} + \ldots$$

$$H_z(y) = H_{zo} + \frac{\partial H_z}{\partial z}(y - y_o) + \frac{\partial^2 H_z}{\partial y^2}\frac{(y - y_o)^2}{2!} + \ldots$$

For the differential surface ΔS, the distances $(y - y_o)$ and $(z - z_o)$ are very small and the higher order terms may be neglected. The approximations for the magnetic field terms are thus

$$H_y(z) \approx H_{yo} + \frac{\partial H_y}{\partial z}(z - z_o)$$

$$H_z(y) \approx H_{zo} + \frac{\partial H_z}{\partial y}(y - y_o)$$

The magnetic field must be evaluated on each of the four segments that make up the closed path L. These magnetic field terms are

On ① $$H_y(0) \approx H_{yo} - \frac{\partial H_y}{\partial z}\frac{\Delta z}{2}$$

On ② $$H_z(\Delta y) \approx H_{zo} + \frac{\partial H_z}{\partial y}\frac{\Delta y}{2}$$

On ③ $$H_y(\Delta z) \approx H_{yo} + \frac{\partial H_y}{\partial z}\frac{\Delta z}{2}$$

On ④ $$H_z(0) \approx H_{zo} - \frac{\partial H_z}{\partial y}\frac{\Delta y}{2}$$

The results of the integrals on the four segments are

$$\int_{①} \boldsymbol{H}\cdot d\boldsymbol{l} + \int_{③} \boldsymbol{H}\cdot d\boldsymbol{l} = \left[H_{yo} - \frac{\partial H_y}{\partial z}\frac{\Delta z}{2}\right]\Delta y - \left[H_{yo} + \frac{\partial H_y}{\partial z}\frac{\Delta z}{2}\right]\Delta y$$

$$\int_{②} \boldsymbol{H}\cdot d\boldsymbol{l} + \int_{④} \boldsymbol{H}\cdot d\boldsymbol{l} = \left[H_{zo} + \frac{\partial H_z}{\partial y}\frac{\Delta y}{2}\right]\Delta z - \left[H_{zo} - \frac{\partial H_z}{\partial y}\frac{\Delta y}{2}\right]\Delta z$$

or

$$\oint \boldsymbol{H}\cdot d\boldsymbol{l} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\Delta y\,\Delta z = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\Delta S$$

The x-component of the curl operator becomes

$$(\boldsymbol{a}_x \cdot \text{curl}\,\boldsymbol{H})\boldsymbol{a}_x \equiv \lim_{\Delta S \to 0} \frac{\oint \boldsymbol{H}\cdot d\boldsymbol{l}}{\Delta S}\boldsymbol{a}_x = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\boldsymbol{a}_x = J_x\boldsymbol{a}_x$$

The y- and z-components of the curl operator may be determined in a similar fashion by applying Ampere's law on differential surfaces that are normal to the y and z directions, respectively.

The results for the y- and z-components of the curl operator are

$$(\boldsymbol{a}_y \cdot \text{curl}\,\boldsymbol{H})\boldsymbol{a}_y \equiv \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \boldsymbol{a}_y = J_y \boldsymbol{a}_y$$

$$(\boldsymbol{a}_z \cdot \text{curl}\,\boldsymbol{H})\boldsymbol{a}_z \equiv \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \boldsymbol{a}_z = J_z \boldsymbol{a}_z$$

The overall curl operator in rectangular coordinates is

$$\text{curl}\boldsymbol{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \boldsymbol{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \boldsymbol{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \boldsymbol{a}_z$$

$$= J_x \boldsymbol{a}_x + J_y \boldsymbol{a}_y + J_z \boldsymbol{a}_z = \boldsymbol{J}$$

Thus, the differential form of Ampere’s law is

$$\text{curl}\boldsymbol{H} = \boldsymbol{J}$$

The curl operator can be written in terms of the gradient operator as

$$\text{curl}\boldsymbol{H} = \nabla \times \boldsymbol{H} = \left(\frac{\partial}{\partial x}\boldsymbol{a}_x + \frac{\partial}{\partial y}\boldsymbol{a}_y + \frac{\partial}{\partial z}\boldsymbol{a}_z \right) \times \left[H_x \boldsymbol{a}_x + H_y \boldsymbol{a}_y + H_z \boldsymbol{a}_z \right]$$

which can also be written in determinant form as

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

The same technique is used to determine the curl operator in cylindrical and spherical coordinates. In these cases, we must use differential surfaces that match the given coordinate system.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Rectangular Coordinates

$$\nabla \times \boldsymbol{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \boldsymbol{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \boldsymbol{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \boldsymbol{a}_z$$

Cylindrical Coordinates

$$\nabla \times \boldsymbol{H} = \left(\frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right) \boldsymbol{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} \right) \boldsymbol{a}_\phi + \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho H_\phi) - \frac{\partial H_\rho}{\partial \phi} \right) \boldsymbol{a}_z$$

Spherical Coordinates

$$\nabla \times \boldsymbol{H} = \frac{1}{r \sin\theta} \left(\frac{\partial}{\partial \theta} (H_\phi \sin\theta) - \frac{\partial H_\theta}{\partial \phi} \right) \boldsymbol{a}_r + \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial H_r}{\partial \phi} - \frac{\partial}{\partial r} (r H_\phi) \right) \boldsymbol{a}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r H_\theta) - \frac{\partial H_r}{\partial \theta} \right) \boldsymbol{a}_\phi$$

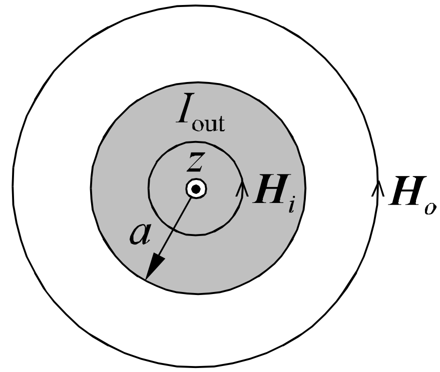
Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Example (Differential form of Ampere's law)

Given the magnetic field inside and outside a conductor of radius a carrying a uniform current density (total current = I_{out}), show that the differential form of Ampere's law yields the current density in both regions.

$$\boldsymbol{H}_i = \frac{I\rho}{2\pi a^2}\boldsymbol{a}_\phi \qquad (\rho < a)$$

$$\boldsymbol{H}_o = \frac{I}{2\pi\rho}\boldsymbol{a}_\phi \qquad (\rho > a)$$



From the differential form of Ampere's law, evaluating the curl (in cylindrical coordinates) of the magnetic field inside and outside the conductor should yield the current density inside and outside the conductor. The current density inside and outside the conductor is

$$\boldsymbol{J}_i = \frac{I}{\pi a^2}\boldsymbol{a}_z \qquad (\rho < a)$$

$$\boldsymbol{J}_o = 0 \qquad (\rho > a)$$

The curl of the magnetic field in cylindrical coordinates is

$$\nabla\times\boldsymbol{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial\phi} - \frac{\partial H_\phi}{\partial z}\right)\boldsymbol{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial\rho}\right)\boldsymbol{a}_\phi + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho H_\phi) - \frac{\partial H_\rho}{\partial\phi}\right)\boldsymbol{a}_z$$

Since the magnetic fields for $r < a$ and $r > a$ have only $\boldsymbol{a}_\phi$ components that are functions of ρ only, all terms in the curl expression are zero except the

first term of the $\boldsymbol{a}_z$ component.
Application of the curl operator yields

$$\nabla \times \boldsymbol{H} = \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho H_\phi)\right)\boldsymbol{a}_z$$

$$= \begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\dfrac{I\rho^2}{2\pi a^2}\right)\boldsymbol{a}_z & (\rho < a) \\ \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\dfrac{I}{2\pi}\right)\boldsymbol{a}_z & (\rho > a) \end{cases}$$

$$= \begin{cases} \dfrac{1}{\rho}\left(\dfrac{I}{2\pi a^2}\right)(2\rho)\,\boldsymbol{a}_z & (\rho < a) \\ \dfrac{1}{\rho}(0)\boldsymbol{a}_z & (\rho > a) \end{cases}$$

$$= \begin{cases} \dfrac{I}{\pi a^2}\boldsymbol{a}_z = \boldsymbol{J}_i & (\rho < a) \\ 0\,\boldsymbol{a}_z = \boldsymbol{J}_o & (\rho > a) \end{cases}$$

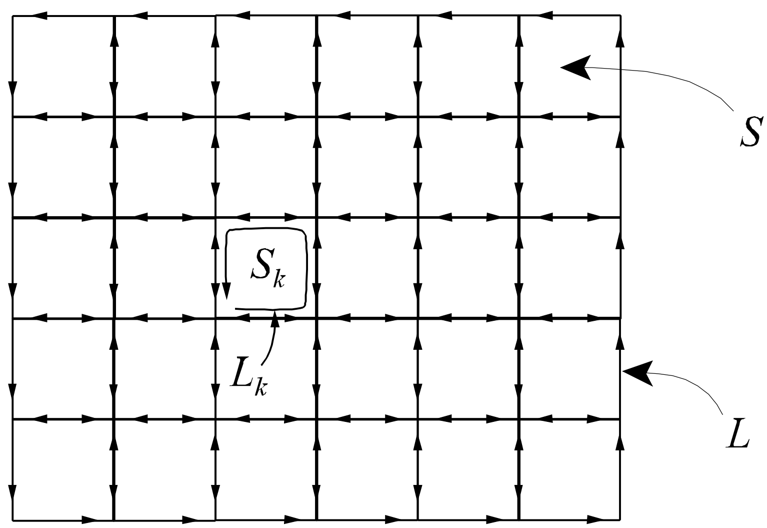
Thus, given the magnetic field everywhere for a particular current distribution, application of Ampere's law in differential form (curl of ***H***) gives the current density that produces the magnetic field.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Stoke's Theorem

Stoke's theorem is a vector identity that defines the transformation of a line integral of a vector around a closed path into a surface integral over the surface bounded by that path. The integrand of the resulting surface integral is the curl of the vector.

Given a surface S bounded by a path L, the surface can be subdivided into cells of surface area ΔS_k bounded by paths L_k. If we apply Ampere's law to each cell and sum the results, the contributions from the internal paths on adjacent cells cancel. The net result is the integral of the magnetic field around the outer path L.



The integrals are related by

$$\oint_L \boldsymbol{H}\cdot d\boldsymbol{l} = \sum_k \oint_{L_k} \boldsymbol{H}\cdot d\boldsymbol{l} = I_{enclosed}$$

or

$$\oint_L \boldsymbol{H}\cdot d\boldsymbol{l} = \sum_k \left[\frac{\oint_{L_k} \boldsymbol{H}\cdot d\boldsymbol{l}}{\Delta S_k} \right] \Delta S_k = I_{enclosed}$$

Taking the limit as ΔS_k approaches zero yields

$$\oint_L \boldsymbol{H}\cdot d\boldsymbol{l} = \sum_k \left\{ \lim_{\Delta S_k \to 0} \left[\frac{\oint_{L_k} \boldsymbol{H}\cdot d\boldsymbol{l}}{\Delta S_k} \right] \right\} \Delta S_k = I_{enclosed}$$

In the limit as ΔS_k approaches zero, the discrete sum becomes a continuous sum (an integral).

$$\sum_k \left\{ \quad \right\} \Delta S_k = \iint_S \left\{ \quad \right\} ds$$

The term in brackets above is the definition of the curl of $\boldsymbol{H}$ in the direction normal to the surface S such that

$$\lim_{\Delta S_k \to 0} \left[\frac{\oint_{L_k} \boldsymbol{H}\cdot d\boldsymbol{l}}{\Delta S_k} \right] = (\nabla \times \boldsymbol{H})\cdot \boldsymbol{a}_n$$

where $\boldsymbol{a}_n$ is the unit normal to the surface S. The Ampere's law integral above becomes

$$\oint_L \boldsymbol{H}\cdot d\boldsymbol{l} = \iint_S (\nabla \times \boldsymbol{H})\cdot d\boldsymbol{s} = I_{enclosed}$$

This integral relationship, shown here in terms of Ampere's law, is actually a vector identity which is valid for any vector $\boldsymbol{F}$ and any surface S.

$$\oint_L \boldsymbol{F}\cdot d\boldsymbol{l} = \iint_S (\nabla \times \boldsymbol{F})\cdot d\boldsymbol{s} \qquad \text{(Stoke's theorem)}$$

Using Stoke's theorem, the integral form of Ampere's law can be directly transformed into the differential form.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = I_{enclosed} = \iint_S \boldsymbol{J} \cdot d\boldsymbol{s}$$

Applying Stoke's theorem gives

$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = \iint_S (\nabla \times \boldsymbol{H}) \cdot d\boldsymbol{s} = \iint_S \boldsymbol{J} \cdot d\boldsymbol{s}$$

Since the surface integrals in this equation are valid for any surface *S*, the integrands of the two integrals must be equal. This yields Ampere's law in differential form $\nabla \times \boldsymbol{H} = \boldsymbol{J}$.

Gauss's Law for Magnetic Fields

Two of the four Maxwell's equations have been defined thus far: Gauss's law (for electric fields) and Ampere's law. The third Maxwell's equation is Gauss's law applied to magnetic fields. The integral and differential forms of Gauss's law for magnetic fields can be determined from the corresponding electric field equations.

Electric fields	Magnetic fields
$\boldsymbol{D} = \epsilon \boldsymbol{E}$	$\boldsymbol{B} = \mu \boldsymbol{H}$
Gauss's Law − electric fields (integral form)	Gauss's Law − magnetic fields (integral form)
$\oiint_S \boldsymbol{D} \cdot d\boldsymbol{s} = Q_{enclosed}$	$\oiint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0$
Gauss's Law − electric fields (differential form)	Gauss's Law − magnetic fields (differential form)
$\nabla \cdot \boldsymbol{D} = \rho_v$	$\nabla \cdot \boldsymbol{B} = 0$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

The charge terms on the right hand side of Gauss's law for magnetic fields (integral and differential form) are zero since the dual parameter to electric charge (magnetic charge) does not exist. The characteristics of electrostatic and magnetostatic fields are fundamentally different based on the existence or nonexistence of charge.

Electrostatic Fields	Magnetostatic Fields
Electric flux lines begin on positive charge and end on negative charge. (Discontinuous)	Magnetic flux lines form closed loops (Continuous)

Faraday's Law for Electrostatic Fields

The last of the four Maxwell's equations is Faraday's law. The general time-varying form of Faraday's law will be discussed when time-varying fields are considered. The electrostatic form of Faraday's law is simply a statement of the conservative nature of the electrostatic field (the closed line integral of the electrostatic field is zero).

$$\oint_L \boldsymbol{E} \cdot d\boldsymbol{l} = 0 \qquad \left(\begin{array}{c}\text{Faraday's law - integral form} \\ \text{electrostatic fields}\end{array}\right)$$

Using Stoke's theorem, this integral can also be written as

$$\oint_L \boldsymbol{E} \cdot d\boldsymbol{l} = \iint_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{s} = 0$$

Since the surface integral above is valid for any surface *S*, the electric field must satisfy

$$\nabla \times \boldsymbol{E} = 0 \qquad \left(\begin{array}{c}\text{Faraday's law - differential form} \\ \text{electrostatic fields}\end{array}\right)$$

which is the differential form of Faraday's law for electrostatic fields.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Maxwell's Equations for Static Fields

All four of Maxwell's equations for static fields have been defined in both integral form and differential form. Maxwell's equations for time-varying fields contain additional terms which form a complete set of coupled equations (all four equations must be satisfied simultaneously). For static fields, Maxwell's equations de-couple into two sets of two equations: two for electrostatic fields and two for magnetostatic fields. Maxwell's equations for static fields are:

	Integral form	Differential form	
Static $\boldsymbol{E}$	$\iint_S \boldsymbol{D}\cdot d\boldsymbol{s} = \iiint_V \rho_v dv = Q_{enclosed}$	$\nabla\cdot\boldsymbol{D} = \rho_v$	①
	$\oint_L \boldsymbol{E}\cdot d\boldsymbol{s} = 0$	$\nabla\times\boldsymbol{E} = 0$	②
Static $\boldsymbol{H}$	$\iint_S \boldsymbol{B}\cdot d\boldsymbol{s} = 0$	$\nabla\cdot\boldsymbol{B} = 0$	③
	$\oint_L \boldsymbol{H}\cdot d\boldsymbol{l} = \iint_S \boldsymbol{J}\cdot d\boldsymbol{s} = I_{enclosed}$	$\nabla\times\boldsymbol{H} = \boldsymbol{J}$	④

① Gauss's law (electric fields)

② Faraday's law

③ Gauss's law (magnetic fields)

④ Ampere's law

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Differential Operators in Electromagnetics

The differential form of the governing equations in electromagnetics (Maxwell's equations and related equations) are defined in terms of four different differential operators: the gradient operator, the divergence operator, the Laplacian operator and the curl operator. All of these operators can be defined in terms of the gradient (∇) operator.

Operators Involving ∇

Operator	Example	Operand	Result
Gradient	$\nabla V = -\boldsymbol{E}$	scalar	vector
Divergence	$\nabla\cdot\boldsymbol{D} = \rho_v$	vector	scalar
Laplacian	$\nabla^2 V = -\dfrac{\rho_v}{\epsilon}$	scalar	scalar
Curl	$\nabla\times\boldsymbol{H} = \boldsymbol{J}$	vector	vector

Note that the two operators that operate on vectors (divergence and curl) are the two operators found in the differential form of Maxwell's equations. Certain characteristics of the vector fields in Maxwell's equations can be determined based on the divergence and curl results for these fields.

Characteristics of $\boldsymbol{F}$ based on $\nabla\cdot\boldsymbol{F}$

Vectors with nonzero divergence ($\nabla\cdot\boldsymbol{F} \neq 0$) vary in the direction of the field.

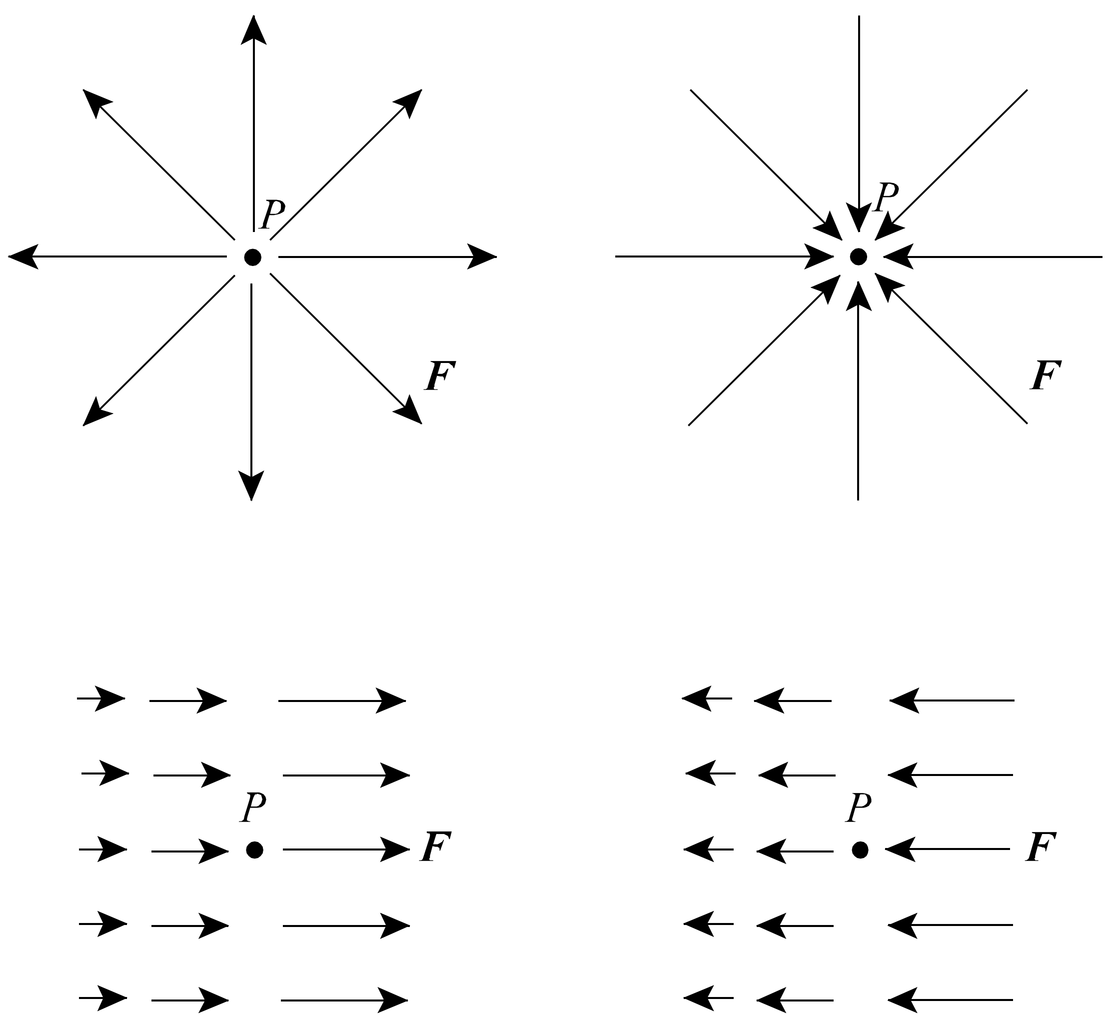
Vectors with zero divergence ($\nabla\cdot\boldsymbol{F} = 0$) do not vary in the direction of the field.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

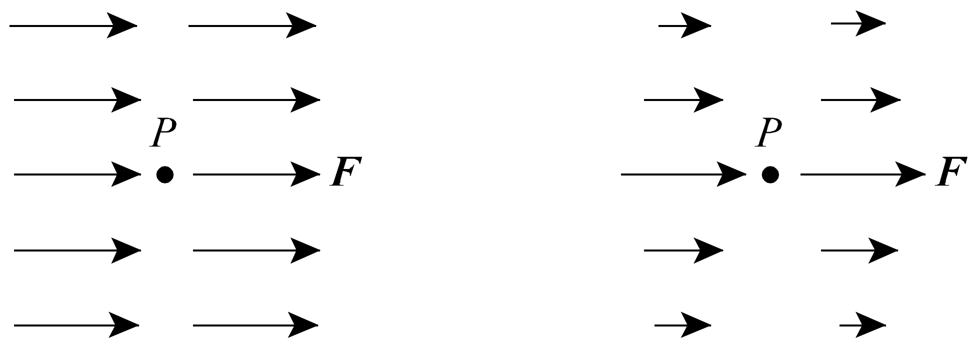
The divergence of a vector $\boldsymbol{F}$ at a point P can be visualized by enclosing the point with an infinitesimally small differential volume and examining the flux of the vector in and out of the volume. If there is a net flux out of the volume (more flux out of the volume than into the volume), the divergence of $\boldsymbol{F}$ is positive at the point P. If there is a net flux into the volume (more flux into the volume than out the volume), the divergence of $\boldsymbol{F}$ is negative at the point P.

$(\nabla \cdot \boldsymbol{F})_P > 0$ $\qquad\qquad$ $(\nabla \cdot \boldsymbol{F})_P < 0$



If the net flux into the differential volume is zero (the flux into the volume equals the flux out of the volume), the divergence of $\boldsymbol{F}$ is zero at P.

$$(\nabla \cdot \boldsymbol{F})_P = 0$$



According to Gauss's law for electric fields in differential form,

$$\nabla \cdot \boldsymbol{D} = \rho_v$$

the divergence of the electric flux density is zero in a charge-free region ($\rho_v = 0$) and non-zero in a region where charge is present. Thus, the divergence of the electric flux density locates the source of the electrostatic field (net positive charge = net flux out, net negative charge = net flux in).

According to Gauss's law for magnetic fields in differential form,

$$\nabla \cdot \boldsymbol{B} = 0$$

the divergence of the magnetic flux density is always zero since there is no magnetic charge (net flux = 0).

Characteristics of $\boldsymbol{F}$ based on $\nabla \times \boldsymbol{F}$

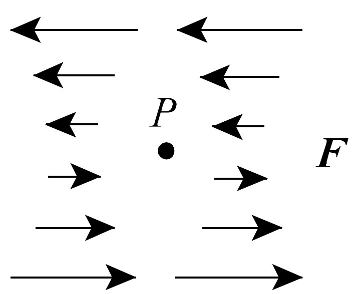
Vectors with nonzero curl ($\nabla \times \boldsymbol{F} \neq 0$) vary in a direction ⊥ to the direction of the field.

Vectors with zero curl ($\nabla \times \boldsymbol{F} = 0$) do not vary in a direction ⊥ to the direction of the field.

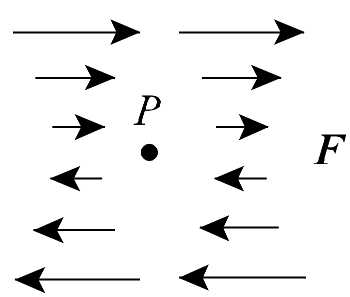
Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

The curl of a vector $\boldsymbol{F}$ at a point P can be visualized by inserting a small paddle wheel into the field (interpreting the vector $\boldsymbol{F}$ as a force field) and noting if the paddle wheel rotates or not. If there is an imbalance of force on the sides of the paddle wheel, the wheel will rotate and the curl of $\boldsymbol{F}$ is in the direction of the wheel axis (according to the right hand rule). If the forces on both sides are equal, there is no rotation, and the curl is zero. The magnitude of the rotation velocity represents the magnitude of the curl of $\boldsymbol{F}$ at P. The curl of the vector field $\boldsymbol{F}$ is therefore a measure of the circulation of $\boldsymbol{F}$ about the point P.

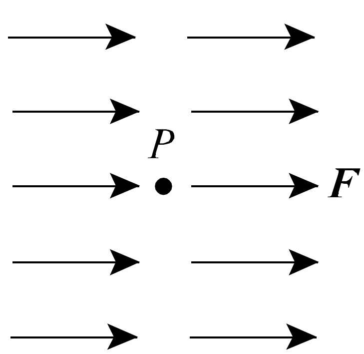
$(\nabla\times\boldsymbol{F})_P \rightarrow$ **out**



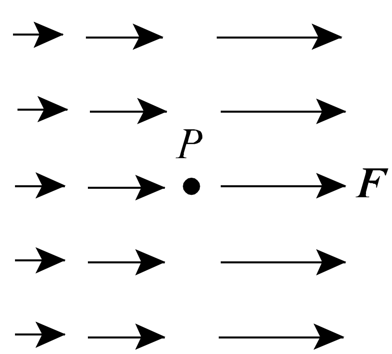
$(\nabla\times\boldsymbol{F})_P \rightarrow$ **in**



$(\nabla\times\boldsymbol{F})_P = 0$



$(\nabla\times\boldsymbol{F})_P = 0$



According to Ampere's law in differential form,

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

the curl of the magnetic field is zero in a current-free region ($\boldsymbol{J} = 0$) and non-zero in a region where current is present. Thus, the curl of the magnetostatic field locates the source of the field (steady current).

According to Faraday's law in differential form,

$$\nabla \times \boldsymbol{E} = 0$$

the curl of the electrostatic field is always zero.

Static Fields and Potentials

Fields with zero curl are defined as *lamellar* or *irrotational* fields. All electrostatic fields are lamellar fields. According to the vector identity,

$$\nabla \times \nabla f = 0$$

electrostatic fields can be written as the gradient of some scalar (electric scalar potential - *V*).

$$\boldsymbol{E} = -\nabla V$$

In a similar fashion, in a current-free region ($\boldsymbol{J}=0$), the magnetic field is lamellar ($\nabla\times\boldsymbol{H}=0$) so that the magnetic field may also be written as the gradient of some scalar.

$$\boldsymbol{H} = -\nabla V_m$$

where V_m is the *magnetic scalar potential*.

Fields with zero divergence are defined as *solenoidal* or *rotational* fields. All magnetostatic fields are solenoidal based on Gauss's law for magnetic fields.

$$\nabla \cdot \boldsymbol{B} = 0$$

According to the vector identity

$$\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$$

magnetostatic fields can be written as the curl of some vector (magnetic vector potential - ***A***). Thus, we may write

$$\nabla \cdot \boldsymbol{B} = 0 \quad \Rightarrow \quad \boldsymbol{B} = \mu \boldsymbol{H} = \nabla \times \boldsymbol{A} \quad \Rightarrow \quad \boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A}$$

Inserting the magnetic field expression into the differential form of Ampere's law gives

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \quad \Rightarrow \quad \nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{A} \right) = \boldsymbol{J} \quad \Rightarrow \quad \nabla \times \nabla \times \boldsymbol{A} = \mu \boldsymbol{J}$$

The curl curl operator satisfies the following vector identity:

$$\nabla \times \nabla \times \boldsymbol{F} = \nabla (\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$$

where the last term in the previous equation is defined as the *vector Laplacian*. The equation defining the magnetic vector potential in terms of the current density becomes

$$\nabla (\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu \boldsymbol{J}$$

We are free to choose the characteristics of the vector potential to simplify the mathematics, so long as the fields defined in terms of ***A*** still satisfy Maxwell's equations. If we choose

$$\nabla \cdot \boldsymbol{A} = 0$$

then the equation for the magnetic vector potential in terms of the current density becomes

$$\nabla^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

This equation is the vector analogy to Poisson's equation:

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

The solution to the magnetic vector potential differential equation takes the same form as the solution to the electric scalar potential differential equation.

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \qquad \Leftrightarrow \qquad \nabla^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

$$V = \frac{1}{4\pi\epsilon} \iiint\limits_V \frac{\rho_v}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d\boldsymbol{v}' \qquad \Leftrightarrow \qquad \boldsymbol{A} = \frac{\mu}{4\pi} \iiint\limits_V \frac{\boldsymbol{J}}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d\boldsymbol{v}'$$

Given the integral for the magnetic vector potential in terms of the current density, the magnetostatic field can be determined by first evaluating the integral in terms of the known current density, then differentiating ***A*** to find ***B*** according to

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

The general 3D integral for the magnetic vector potential in terms of the volume current density can be simplified for surface or line currents.

$$\boldsymbol{A} = \frac{\mu}{4\pi} \iiint\limits_V \frac{\boldsymbol{J}}{|\boldsymbol{r}-\boldsymbol{r}'|}\, dv' \qquad \text{(volume currents)}$$

$$\boldsymbol{A} = \frac{\mu}{4\pi} \iint\limits_S \frac{\boldsymbol{K}}{|\boldsymbol{r}-\boldsymbol{r}'|}\, ds' \qquad \text{(surface currents)}$$

$$\boldsymbol{A} = \frac{\mu}{4\pi} \int\limits_L \frac{\boldsymbol{I}}{|\boldsymbol{r}-\boldsymbol{r}'|}\, dl' \qquad \text{(line currents)}$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Example (Vector potential)

Given a magnetic vector potential of

$$\boldsymbol{A} = e^{-z}\cos y\,\boldsymbol{a}_x + (1 + \sin x)\,\boldsymbol{a}_z \qquad (\text{Wb/m})$$

determine (a.) the vector magnetic flux density $\boldsymbol{B}$ and (b.) the total magnetic flux ψ_m passing through a square loop defined by $(0<x<\pi)$, $(0<y<\pi)$ and $z=0$.

(a.) $\boldsymbol{B} = \nabla\times\boldsymbol{A}$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{a}_z$$

$$= [(-e^{-z}\cos y) - (\cos x)]\,\boldsymbol{a}_y + [-(-e^{-z}\sin y)]\,\boldsymbol{a}_z$$

$$= -(e^{-z}\cos y + \cos x)\,\boldsymbol{a}_y + (e^{-z}\sin y)\,\boldsymbol{a}_z \qquad (\text{Wb/m}^2)$$

(b.)

$$\psi_m = \iint_S \boldsymbol{B}\cdot d\boldsymbol{s}$$

$$= \int_0^{\pi}\int_0^{\pi} B_z(z=0)\,dx\,dy$$

$$= \int_0^{\pi}\int_0^{\pi} (\sin y)\,dx\,dy$$

$$= [x]_0^{\pi}\,[-\cos y]_0^{\pi}$$

$$= \pi[1-(-1)] = 2\pi = 6.28\ \text{Wb}$$