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Chapter 2

System Modelling

2.1 Introduction

If the dynamic behavior of a physical system can be represented by an equation, or a set of equations, this is referred to as the mathematical model of the system. Such models can be constructed from knowledge of the physical characteristics of the system, i.e. mass for a mechanical system or resistance for an electrical system. Because the systems under consideration are dynamic in nature, the descriptive equations are usually *differential equations*. Differential equations are often the initial description of a system. The variables are just the inputs and outputs. If the differential equations can be linearized, then the Laplace transform can be utilized to simplify the method of solution. a single-input single-output process is described by its *transfer function*: the ratio of the Laplace transform of output and input.

2.2 The Laplace transform

The Laplace transform is a very powerful analysis tool for a certain class of systems, namely, linear time-invariant systems. It transforms the problem from the time (or t) domain to the Laplace (or s) domain. The advantage in doing this is that complex time-domain differential equations become relatively simple s-domain algebraic equations. It is then possible to manipulate the algebraic equation by simple algebraic rules to obtain the solution in the s-domain. When a suitable solution is arrived at, it is inverse transformed back to the time-domain.

2.2.1 Definition

The Laplace transform of a function of time $f(t)$, $0 \leq t \leq \infty$ with $f(t) = 0$ for $t \leq 0$ is defined as

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)e^{-st}dt \tag{2.1}$$

where s is a complex variable $s = \sigma + j\omega$ and is called the laplace operator.

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Example 2.1 Let $f(t)$ be a unit step function defined as $f(t) = 1$ for $t \geq 0$.

■ **Solution** The Laplace transform of $f(t)$ is obtained as

$$F(s) = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s} \qquad \blacksquare$$

Example 2.2 Consider the exponential function $f(t) = e^{-at}$ for $t \geq 0$.

■ **Solution** The Laplace transform of $f(t)$ is

$$F(s) = \int_0^\infty e^{-at} e^{-st} dt = -\frac{e^{-(s+a)t}}{s+a} \Big|_0^\infty = \frac{1}{s+a} \qquad \blacksquare$$

2.2.2 Properties

The application of the Laplace transform in many instances is simplified by utilization of the properties of the transform. These properties are presented here, for which no proofs are given.

Linearity

$$\mathcal{L}[k_1 f_1(t) \pm k_2 f_2(t)] = k_1 F_1(s) \pm k_2 F_2(s)$$

Differentiation

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

where $f(0)$ is the limit of $f(t)$ as t approaches 0. In general, for higher-order derivatives of $f(t)$,

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$$

where $f^{(i)}(0)$ denotes the ith-order derivative of $f(t)$ with respect to t, evaluated at $t = 0$.

Integration

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}$$

Shift in time

$$\mathcal{L}[f(t-T)] = e^{-Ts} F(s) \qquad \text{for } T \geq 0$$

Shift in frequency

$$\mathcal{L}[e^{\mp \alpha t} f(t)] = F(s \pm \alpha)$$

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COVOLUTION

$$\mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)$$

2.2.3 THEOREMS

INITIAL VALUE THEOREM

A useful property of the Laplace transform known as the initial value theorem which states that it is always possible to determine the initial value of the time function $f(t)$ from its Lapalce transform. We may state the theorem in this way:

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

FINAL VALUE THEOREM [MORE ON THIS PROPERTY LATER]

A second valuable Laplace transform theorem is the final value theorem, it allows us to compute the constant steady-state value of a time function given its Laplace transform.

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

2.2.4 MORE EXAMPLES

In this section more examples are given to demonstrate the utilization of the Laplace transform properties. For the subjects treated in this course, the direct evaluation of the Lapalce transform integral is almost never used.

Find the Laplace transform of $f(t) = \cos \omega t$. [Linearity property] **Example 2.3**

■ **Solution** The Laplace transform is

$$F(s) = \int_0^\infty (\cos \omega t) e^{-st} dt$$

We substitute the relation

$$\cos \omega t = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

into the integral, we find that

$$\begin{aligned} F(s) &= \frac{1}{2}\left(\frac{1}{s - j\omega}\right) + \frac{1}{2}\left(\frac{1}{s + j\omega}\right) \\ &= \frac{s}{s^2 + \omega^2} \quad \blacksquare \end{aligned}$$

Find the Laplace transform of $f(t) = \dfrac{d^2 f}{dt^2}$. [Differentiation property] **Example 2.4**

■ **Solution** The Laplace transform is

$$F(s) = s^2 F(s) - sf(0) - \left.\frac{df}{dt}\right|_{t=0} \quad \blacksquare$$

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Example 2.5 Find the Laplace transform of $f(t) = e^{-\alpha t}\cos\omega t$. [Shifting in frequency]

■ **Solution** From Example 2.3 we know that

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}$$
$$\Longrightarrow \mathcal{L}[e^{-\alpha t}\cos\omega t] = \frac{s+\alpha}{(s+\alpha)^2+\omega^2} \quad \blacksquare$$

Table 2.1: Common Laplace transform pairs

	Time function $f(t)$	*Laplace transform* $\mathcal{L}[f(t)] = F(s)$
1.	unit impulse $\delta(t)$	1
2.	unit step $u(t)$	$\frac{1}{s}$
3.	unit ramp t	$\frac{1}{s^2}$
4.	t^n	$\frac{n!}{s^{n+1}}$
5.	e^{-at}	$\frac{1}{s+a}$
6.	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
7.	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
8.	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
9.	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
10.	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

2.2.5 Inverse Laplace transform

The inverse transform of a function of s is given by the integral

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}ds$$

In general, this expression is difficult to evaluate. In practice, inverse transformation is most easily achieved by using partial fractions to break down solutions into standard components, and then use tables of Laplace transform pairs, as given in Table 2.1.

2.2.6 Partial fraction expansion

A rational function $F(s)$ can be written as

$$F(s) = \frac{Q(s)}{P(s)}$$

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where $P(s)$ and $Q(s)$ are polynomials of s. Rational functions are defined as the ratio of two polynomials. It is assumed that the *order of $P(s)$ in s is greater than that of $Q(s)$, $F(s)$ is said to be strictly proper.* The polynomial $P(s)$ may be written

$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

where $a_0, a_1, \cdots, a_{n-1}$ are real coefficients. The roots of the polynomial $P(s)$ are referred to as *poles* of the function $F(s)$.

CASE 1: $F(s)$ HAS DISTINCT REAL POLES

If all the poles of $F(s)$ are real but distinct, $F(s)$ can be written as

$$F(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_n)}$$

which can be rewritten as a partial-fraction expansion

$$F(s) = \frac{K_1}{s+s_1} + \frac{K_2}{s+s_2} + \cdots + \frac{K_n}{s+s_n}$$

To be able to determine the coefficients K_i $(i = 1, 2, \cdots, n)$ we use the so-called *cover-up method*

$$K_i = (s+s_i)F(s)\Big|_{s=-s_i}$$

Example 2.6

Find the inverse Laplace transform of

$$F(s) = \frac{s+2}{s^3+4s^2+3s}$$

■ **Solution** We may write $F(s)$ as

$$F(s) = \frac{s+2}{s(s+1)(s+3)}$$

and in terms of its partial-fraction expansion:

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+3}$$

Using the cover-up method, we get

$$K_1 = sF(s)\Big|_{s=0} = \frac{s+2}{(s+1)(s+3)}\Big|_{s=0} = \frac{2}{3}$$

In a similar fashion

$$K_2 = (s+1)F(s)\Big|_{s=-1} = \frac{s+2}{s(s+3)}\Big|_{s=-1} = -\frac{1}{2}$$

and

$$K_3 = (s+3)F(s)\Big|_{s=-3} = \frac{s+2}{s(s+1)}\Big|_{s=-3} = -\frac{1}{6}$$

With the partial fraction the solution can be looked up in the tables at once to be

$$f(t) = \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t} \qquad ■$$

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CASE 2: $F(s)$ HAS DISTINCT COMPLEX POLES

We have to take special care of the quadratic factors in the denominator. The numerator of the quadratic factor is chosen to be first-order as shown in the following example.

Example 2.7 Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2+s+1)}$$

■ **Solution** We rewrite $F(s)$ as

$$F(s) = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+s+1}$$

Using the cover-up method, we find K_1 to be

$$K_1 = sF(s)\Big|_{s=0} = 1$$

We equate the numerators,

$$(s^2+s+1) + (K_2 s + K_3)s = 1$$

After equating like powers of s on the two sides of this equation, we find that $K_2 = -1$ and $K_3 = -1$. To make it more suitable for using the tables we use the method of completing the squares to rewrite the partial fraction as

$$F(s) = \frac{1}{s} - \frac{s+\frac{1}{2}+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}$$

From the tables we have,

$$f(t) = 1 - e^{-t/2}\cos\sqrt{\tfrac{3}{4}}t - \frac{1}{\sqrt{3}}e^{-t/2}\sin\sqrt{\tfrac{3}{4}}t$$

CASE 3: $F(s)$ HAS MULTIPLE-ORDER POLES

If r of the n poles of $F(s)$ are identical, or we say that the pole at $s = -s_i$ is of muliplicity r, $F(s)$ is written

$$F(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_{n-r})(s+s_i)^r}$$

$(i \neq 1, 2, \cdots, n-r)$. Then $F(s)$ can be expanded as

$$F(s) = \underbrace{\frac{K_1}{s+s_1} + \frac{K_2}{s+s_2} + \cdots + \frac{K_{n-r}}{s+s_{n-r}}}_{n-r \text{ terms of distinct poles}} + \underbrace{\frac{A_1}{s+s_i} + \frac{A_2}{(s+s_i)^2} + \cdots + \frac{A_r}{(s+s_i)^r}}_{r \text{ terms of repeated poles}}$$

The $(n-r)$ coefficients, K_1, $K_2, \cdots, K_{n-r}$, which corresponds to the distinct poles, may be evaluated by the cover-up method. In, general, we may compute A_k for a factor with multiplicity r as

$$A_{r-k} = \frac{1}{k!}\frac{d^k}{ds^k}\Big[(s+s_i)^r F(s)\Big]\Bigg|_{s=-s_i} \qquad k = 0, 1, \cdots, r-1$$

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Example 2.8

Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s+1)^3(s+2)}$$

■ **Solution** We rewrite the partial fraction as

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)^3}$$

The coefficients corresponding to the distinct poles are

$$K_1 = sF(s)\Big|_{s=0} = \frac{1}{2}$$
$$K_2 = (s+2)F(s)\Big|_{s=-2} = \frac{1}{2}$$

and those of the third-order pole are

$$A_3 = (s+1)^3F(s)\Big|_{s=-1} = -1$$
$$A_2 = \frac{d}{ds}\left[(s+1)^3F(s)\right]\Big|_{s=-1} = 0$$
$$A_1 = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+1)^3F(s)\right]\Big|_{s=-1} = -1$$

The completed partial-fraction expansion is

$$F(s) = \frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1} - \frac{1}{(s+1)^3}$$

The function $f(t)$ is

$$f(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - \frac{1}{2}t^2e^{-t} \qquad ■$$

2.3 Transfer Functions

The classical way of modelling linear time-invariant systems is to use *transfer functions* to represent input-output relations between variables. A transfer function is nothing more than the s plane representation of a physical system that can be described by an ordinary differential equation with constant coefficients.

The transfer function of a linear time-invariant system is the ratio of the Laplace transform of the output to the Laplace transform of the input, with all initial conditions assumed to be zero

$$H(s) = \frac{Y(s)}{X(s)}$$

The transfer function of a system represents the relationship describing the dynamics of the system under consideration. Another way of defining the transfer

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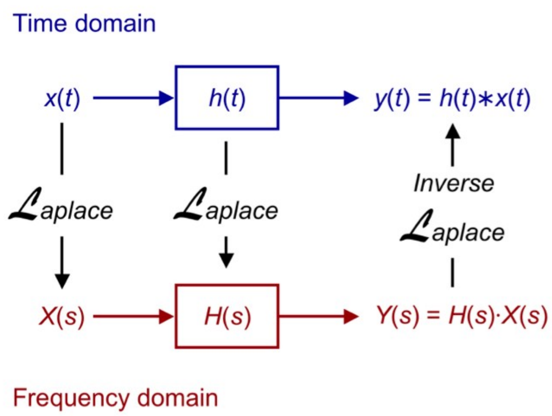

Figure 2.1: The transfer function.

function is to use the impulse response. *The transfer function of a linear time-invariant systems is defined as the Laplace transform of the impulse response, with all initial conditions set to zero*

$$H(s) = \mathcal{L}[h(t)] = \frac{Y(s)}{X(s)}$$

Example 2.9 Find the transfer function of the system described by the following differential equation:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 5$$

with initial conditions

- $y(0) = 4$ and $\dot{y}(0) = 3$

■ **Solution** Take Laplace transform and set all initial conditions to zero

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{5}{s}$$

$$Y(s) = \frac{5}{s(s^2 + 3s + 2)}$$

which can be rearranged as a ratio of the output to the input

$$H(s) = \frac{Y(s)}{(5/s)} = \frac{1}{s^2 + 3s + 2} \quad ■$$

2.4 Models of Electric Circuits

In this section we develop models for simple electrical circuits. We know that for a resistor, the voltage-current relationship in the time domain is

$$v(t) = Ri(t)$$

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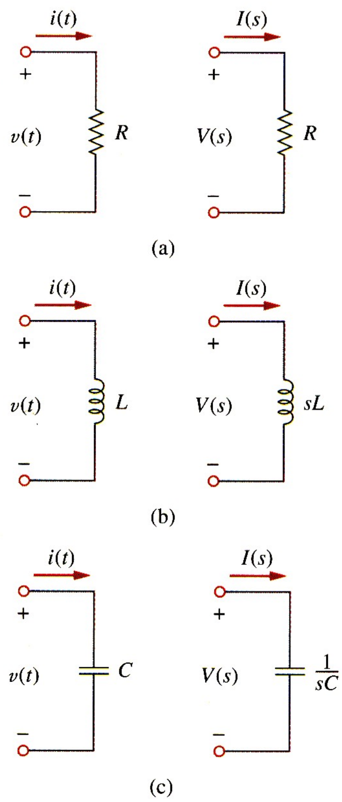

Figure 2.2: Time-domain and s-domain representation of passive elements under zero initial conditions.

Taking Laplace transform, we get

$$V(s) = RI(s)$$

For an inductor,

$$v(t) = L\frac{di(t)}{dt}$$

Taking Laplace and assuming zero initial conditions

$$V(s) = sLI(s)$$

For a capacitor,

$$i(t) = C\frac{dv(t)}{dt}$$

which transforms into the s-domain (assuming zero initial conditions) as

$$V(s) = \frac{1}{sC}I(s)$$

The s-domain equivalents are shown in Figure 2.2.

Example 2.10 Determine the transfer function $H(s) = V_o(s)/I_o(s)$ for the circuit shown in

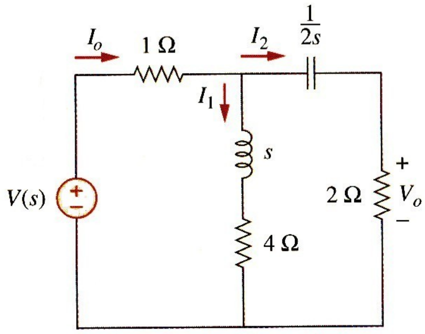

Figure 2.3: For Example 2.10.

Figure 2.3.

■ **Solution** By current division,

$$I_2 = \frac{(s+4)I_o}{s+4+2+\frac{1}{2s}}$$

But

$$V_o = 2I_2 = \frac{2(s+4)I_o}{s+6+\frac{1}{2s}}$$

Hence,

$$H(s) = \frac{V_o(s)}{I_o(s)} = \frac{4s(s+4)}{2s^2+12s+1} \quad \blacksquare$$

Example 2.11 For the circuit shown in Figure 2.4, find the transfer function $I_2(s)/V_(s)$.

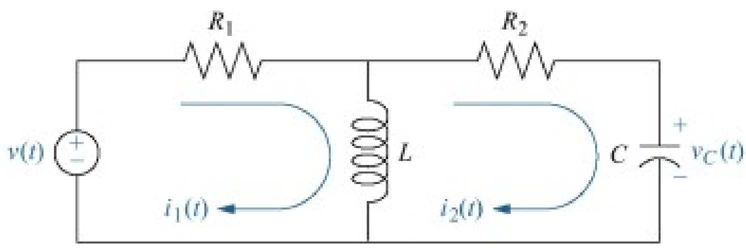

Figure 2.4: Two loop network for Example 2.11.

■ **Solution** The first step in the solution is to convert the network into Laplace transform for impedance and circuit variables, assuming zero initial conditions, as shown in Figure 2.5. The circuit requires two simultaneous equations to solve for the transfer function. These equations can be found by summing voltages around each mesh through which the assumed currents $I_1(s)$ and $I_2(s)$ flow.

Around Mesh 1,

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$

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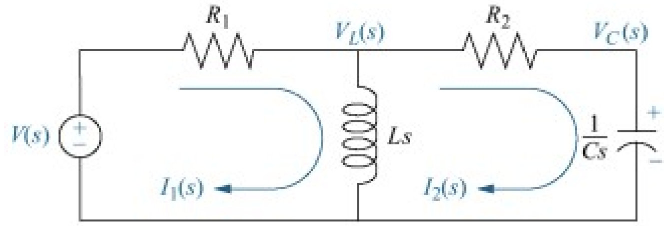

Figure 2.5: s-domain representation of the circuit in Figure 2.4.

Around Mesh 2,

$$LsI_2(s) + R_2I_2(s) + \frac{1}{Cs}I_2(s) - LsI_1(s) = 0$$

By solving the two equations, we get

$$H(s) = \frac{I_2(s)}{V(s)} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

as shown in Figure 2.6. ■

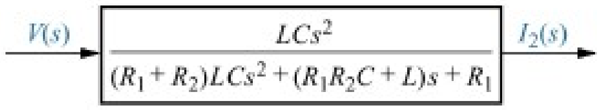

Figure 2.6: The transfer function.

2.5 Models of mechanical systems

We have shown that electrical networks can be modeled by transfer function that algebraically relates the Laplace transform of the output to the Laplace transform of the input. Now, we will do the same for mechanical systems.

The motion of mechanical elements can be described in various dimensions as translational, rotational, or combined. The equations governing the motion of mechanical systems are often formulated directly or indirectly from Newton's laws of motion.

Mechanical systems require one or more differential equations, called the equations of motion, to describe it. We will begin by assuming a positive direction of motion, for example, to the right. This assumed positive direction of motion is similar to assuming a current direction in an electrical loop. Using our assumed direction of positive motion, we first draw a free-body diagram, placing on the body all forces that act on the body either in the direction of motion or opposite to it. Next, we use Newton?s law to form a differential equation of motion by summing the forces and setting the sum equal to zero. Finally, assuming zero initial conditions, we take the Laplace transform of the differential equation, separate the variables and arrive at the transfer function.

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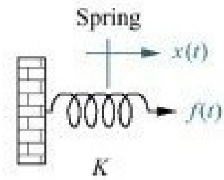
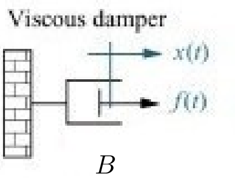
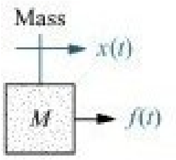
2.5.1 Translational motion

The motion of translation is defined as a motion that takes place along straight lines. The variables that are used to describe translational motion are acceleration, velocity, and displacement. Newton's law states that the algebraic sum of forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction. Table 2.2 shows force-displacement translational relationship for spring, viscous damper and mass. the constants K, B, and M are called spring constant, coefficient of viscous friction, and mass, respectively.

Table 2.2: Force displacement translational relationship for spring, viscous damper and mass.

Component	*Force-displacement*
	$f(t) = Kx(t)$
	$f(t) = B\dfrac{dx(t)}{dt}$
	$f(t) = M\dfrac{d^2x(t)}{dt^2}$

Example 2.12 Find the transfer function $X(s)/F_(s)$ for the system shown in Figure 2.7(a).

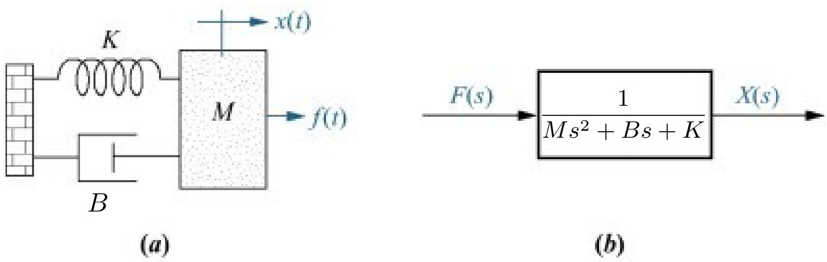


Figure 2.7: (a) Mass, spring and damper system; (b) block diagram.

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■ **Solution** Begin the solution by assuming a positive direction of motion. Then, draw the free-body diagram as shown in Figure 2.8(a). Place on the mass all forces felt by the mass. Only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper and the mass due to acceleration point to the left. We write

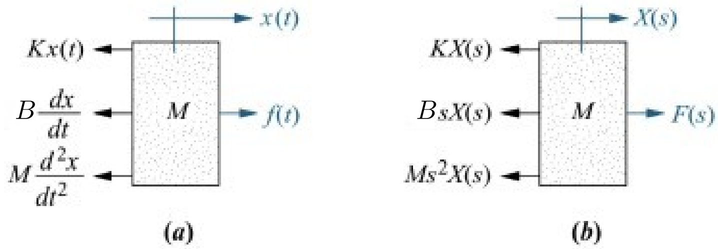

Figure 2.8: (a) Free-body diagram of mass, spring and damper system; (b) transformed free-body diagram (in s domain).

the differential equation of motion using Newton's law to sum to zero all of the forces shown on the mass in Figure 2.8(a):

$$M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t) = f(t)$$

Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + BsX(s) + KX(s) = F(s)$$

which is represented in Figure 2.8(b). Solving for the transfer function yields

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

which is represented in Figure 2.7(b). ■

2.5.2 Rotational motion

Rotational mechanical systems are handled the same way as translational mechanical systems, except that torque replaces force and angular displacement replaces translational displacement. The mechanical components for rotational systems are the same as those for translational systems, except that the components undergo rotation instead of translation. The rotational motion of a body can be defined as motion about a fixed axis. The extension of Newton's law of motion for rotational motion states that the algebraic sum of moments or torque about a fixed axis is equal to the applied angular force or the product of the inertia and the angular acceleration about the axis. Table 2.3 shows the components along with the relationships between torque and angular velocity, as well as angular displacement.
The constants K, D (or sometimes denoted as B) and J are called spring constant, coefficient of viscous friction, and moment of inertia, respectively.

The rotational system shown in Figure 2.9 consists of a disk mounted **Example 2.13**

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Table 2.3: Torqur-angular displacement relationship for spring, viscous damper and inertia.

Component	*Torque-angular displacement*
	$T(t) = K\theta(t)$
	$T(t) = D\dfrac{d\theta(t)}{dt}$
	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$

shaft that is fixed at one end. The moment of inertia of the disk about the axis of rotation is J. The edge of the disk is riding on the surface, and the viscous friction coefficient between the two surfaces is B. The inertia of the shaft is negligible, but the torsional spring constant is K. Assume the torque is applied to the disk, as shown; then the torque or moment of equation about the axis of the shaft is written from the free-body diagram of Figure 2.9(b) as

$$T(t) = J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} + K\theta(t) \qquad \blacksquare \tag{2.2}$$

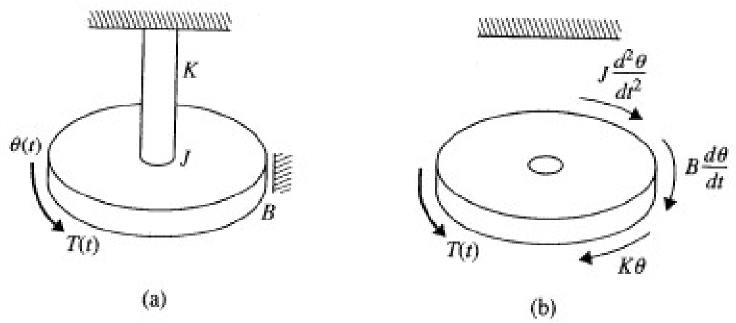

Figure 2.9: Rotational system for Example 2.13.

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2.6 DC motors in control systems

A common actuator in control systems is a DC motor. A motor is an electromechanical component that yields a displacement output for a voltage input. We will derive the transfer function for one particular kind of electromechanical system, the armature-controlled dc servomotor. The motor's schematic is shown in Figure 2.10

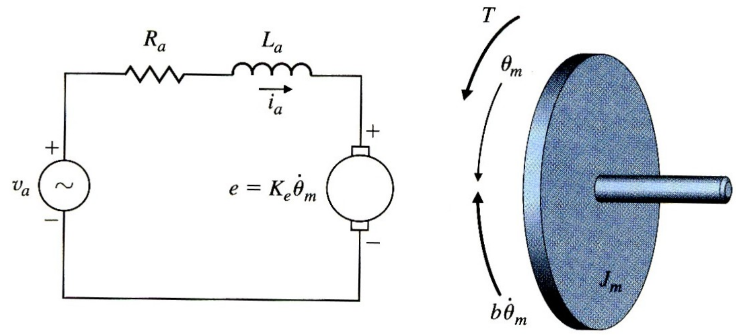

Figure 2.10: A DC motor schematic.

The armature-controlled DC motor uses the armature current i_a as the control variable. In our analysis we need to include the back emf for the electrical circuit. For the mechanical part of the system we need to include the motor torque in analyzing the rotor. Since the armature is rotating in a magnetic field, a back electromotive force[1] (*back emf*), e, is generated

$$e = K_e\dot{\theta}_m \tag{2.3}$$

where K_e is a constant of proprtionality called the back emf constant and $\dot{\theta}_m = \omega_m$ is the angular velocity of the motor. Taking the Laplace transform we have

$$E(s) = K_e s\theta(s) \tag{2.4}$$

The relationship among the armature current i_a, the applied armature voltage v_a, and the back emf e is found by writing a loop equation around the Laplace transformed armature circuit:

$$R_a I_a(s) + L_a s I_a(s) + E(s) = V_a(s) \tag{2.5}$$

The torque developed by the motor is proprtional to the armature current, thus

$$T_m(s) = K_m I_a(s) \tag{2.6}$$

where T_m is the torque developed by the motor and K_m is a constant of proportionality, called the motor-torque constant. Rearranging Equation (2.14) yields

$$I_a(s) = \frac{T_m(s)}{K_m} \tag{2.7}$$

[1]Because the generated electromotive force (emf) works against the applied armature voltage, we call it the *back emf.*

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To find the transfer function of the motor, we first substitute Equation (2.12) and Equation (2.7) into Equation (2.13), yielding

$$\frac{(R_a + L_a s)T_m(s)}{K_m} + K_e s\theta(s) = V_a(s) \tag{2.8}$$

Now we must find $T_m(s)$ in terms of $\theta_m(s)$ if we are to seperate the input and output variables, and to obtain the transfer function $\frac{\theta_m(s)}{V_a(s)}$. The free-body diagram for the rotor, shown in Figure 2.10, defines the positive direction and shows the two applied torques, T and $b\dot{\theta}_m$. Therefore,

$$(J_m s^2 + bs)\theta_m(s) = T_m(s) \tag{2.9}$$

Substituting Equation (2.9) into Equation (2.8) yiields

$$\frac{(R_a + L_a s)(J_m s^2 + bs)\theta_m(s)}{K_m} + K_e s\theta(s) = V_a(s) \tag{2.10}$$

If we assume that the armature inductance L_a is small compared to the armature resistance R_a, which is usually the case for a dc motor, Equation (2.10) becomes

$$\left[\frac{R_a(J_m s + b)}{K_m} + K_e\right] s\theta_m(s) = V_a(s)$$

After simplification, the desired transfer function $\frac{\theta_m(s)}{V_a(s)}$ is found to be

$$\begin{aligned}\frac{\theta_m(s)}{V_a(s)} &= \frac{K_m}{s\left[R_a(J_m s + b) + K_m K_e\right]}\\ &= \frac{K_m}{s\left[J_m R_a s + bR_a + K_m K_e\right]}\end{aligned}$$

The relations of the armature-controlled DC motor are shown schematically in Figure 2.11

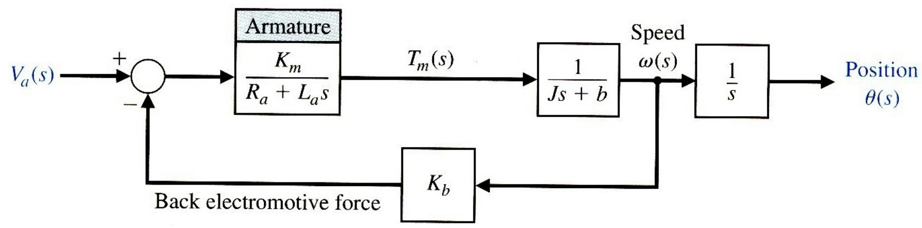

Figure 2.11: Armature-controlled DC motor.

2.7 System modelling diagrams

2.7.1 The block diagram

Block diagrams may be considered as a form of system description that provides a simplified overview schematic diagram of a system. It describes the

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composition and interconnection of a system, or it can be used together with the transfer functions to describe the cause-and-effect relationships throughout the system. The transfer function of each component is placed in a box, and the input-output relationships between components are indicated by lines and arrows. We can then solve the equations by graphical simplification, which is often easier and more informative than algebraic manipulation.

Many practical control systems consist of complicated interconnection of smaller subsystems. Before tackling a control system design for such systems, it is usually helpful to simplify the complex interconnection of subsystems. Essentially, we seek a systematic way to eliminate variables (signals) we do not want to control or measure.

Block diagram algebra

Block diagrams usually consist of (see Figure 2.12):

1. Blocks: these give a description of subsystem dynamics.
2. Summers: add or subtract two or more signals.
3. Arrows: these give the direction of signal propagation.
4. Take off points.

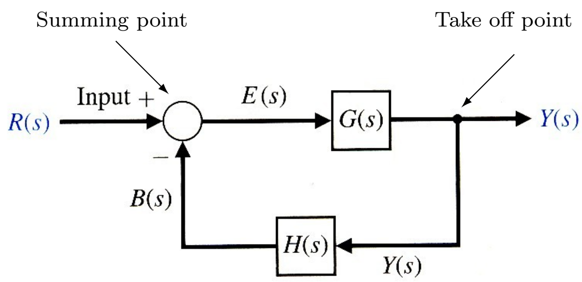

Figure 2.12: Block diagram of a closed-loop system.

Block diagram transformations and reduction techniques are derived by considering the algebra of the diagram variables. For example, consider the block diagram shown in Figure 2.12. This negative feedback control system is described by the equation for the error signal, which is

$$E(s) = R(s) - B(s) = R(s) - H(s)Y(s)$$

Because the output is related to the error signal by $G(s)$, we have

$$Y(s) = G(s)E(s)$$

thus,

$$Y(s) = G(s)[R(s) - H(s)Y(s)]$$

Solving for $Y(s)$, we obtain

$$Y(s)[1 + G(s)H(s)] = G(s)R(s)$$

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Therefore, the transfer function relating the output $Y(s)$ to the input $R(s)$ is

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The gain of a single-loop negative feedback system is given by the forward gain divided by the sum of one plus the loop gain. When the feedback is added instead of subtracted, we call it *positive feedback*. In this case the gain is given by the forward gain divided by the sum of 1 minus the loop gain.

A control system may have several feedback control loops as the one shown in Figure 2.13. In principle, the block diagram of a closed-loop system, no matter how complicated it is, it can be reduced to the standard single loop form shown in Figure 2.12. Reduction of complex block diagrams is facilitated by a series of easily derivable transformations which are summarized in Table 2.4.

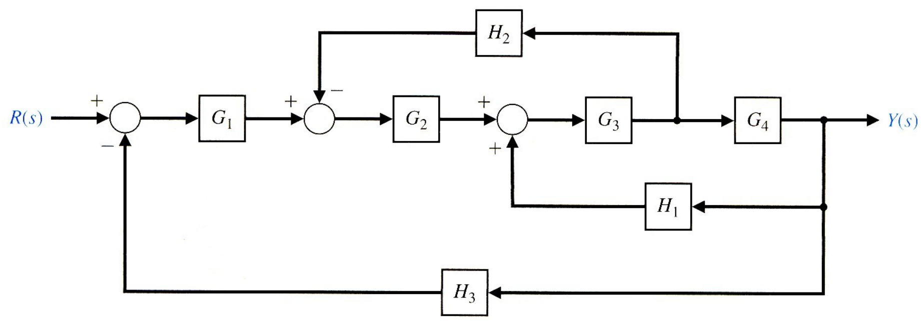

Figure 2.13: Block diagram of a closed-loop system.

The following steps may be used to simplify complicated block diagrams:

1. Combine all cascade blocks.
2. Combine all parallel blocks.
3. Eliminate all minor (interior) feedback loops.
4. Shift summing points to left.
5. Shift take off points to the right.
6. Repeat steps 1 to 5 if necessary.

Block diagram transformations will be illustrated by examples using block diagram reduction.

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Example 2.14

A block diagram of a multiple-loop feedback control system is shown in Figure 2.13. It is interesting to note that the feedback signal $H_1(s)Y(s)$ is a positive feedback signal, and the loop $G_3(s)G_4(s)H_1(s)$ is a positive feedback loop. First, to eliminate the minor loop $G_3G_4H_4$, we move H_2 behind block G_4 by using rule 10 (see Table 2.4), and therefore obtain Figure 2.14.

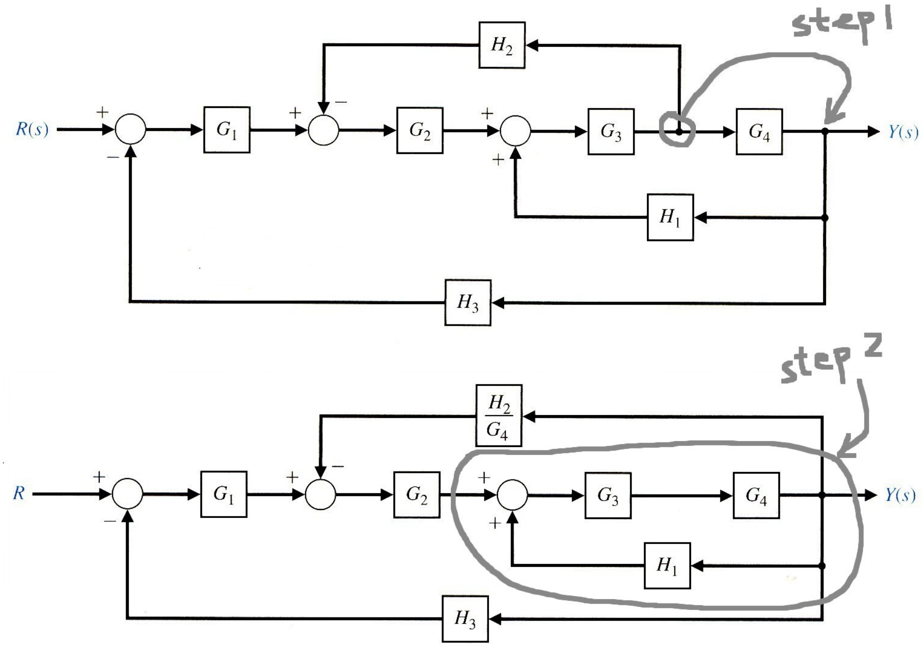

Figure 2.14: Block diagram reduction of the system of Figure 2.13.

Eliminating the loop $G_3G_4H_1$ by using rule 4, we obtain Fogure 2.15.

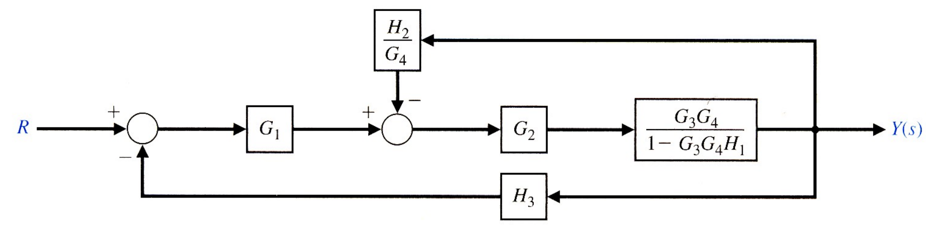

Figure 2.15: Block diagram simplification.

Then eliminating the inner loop containing H_2/G_4, we obtain Figure 2.16(a). Finally, by reducing the loop containing H_3, we obtain the closed-loop system transfer function as shown in Figure 2.16(b).

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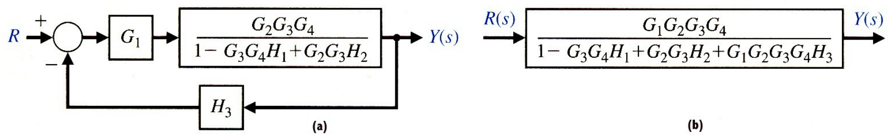

Figure 2.16: Reduced block diagram.

Example 2.15 Find the transfer function of the system shown in Figure 2.17.

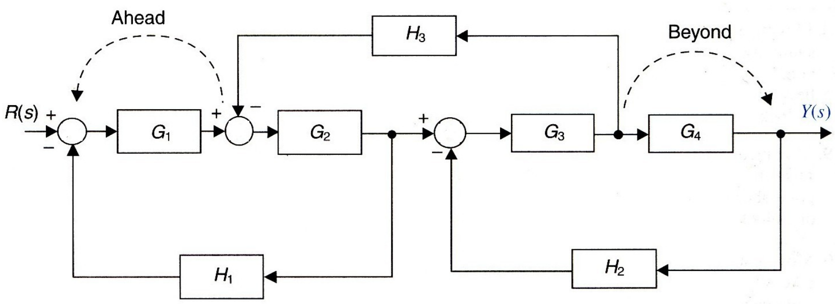

Figure 2.17: Block diagram of Example 2.15.

■ **Solution** Moving the first summing point ahead of G_1, and the final take off point beyond G_4 gives a modified block diagram shown in Figure 2.18(a). The block diagram in Figure 2.18(a) is then reduced to the form given in Figure 2.18(b).

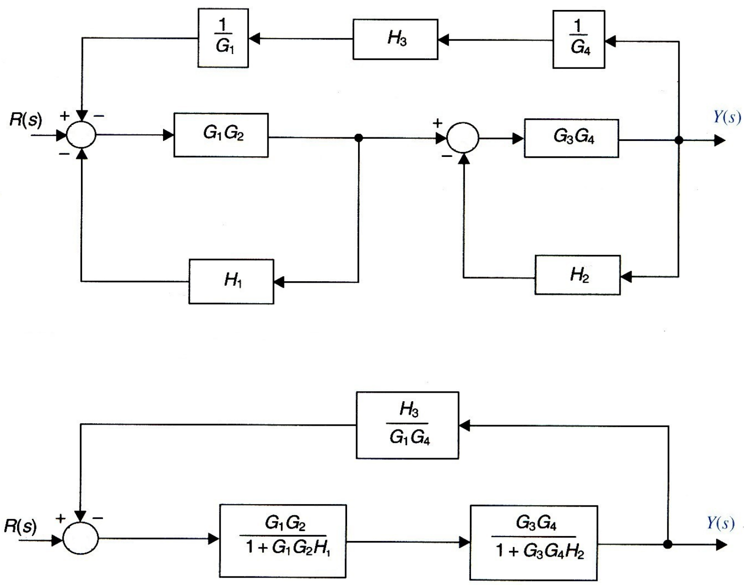

Figure 2.18: Stages of block diagram reduction.

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The overall closed-loop transfer function is then

$$\frac{Y(s)}{R(s)} = \frac{\frac{G_1G_2G_3G_4}{(1+G_1G_2H_1)(1+G_3G_4H_2)}}{1+\frac{G_1G_2G_3G_4H_3}{(G_1G_4)(1+G_1G_2H_1)(1+G_3G_4H_2)}}$$
$$= \frac{G_1G_2G_3G_4}{(1+G_1G_2H_1)(1+G_3G_4H_2)+G_2G_3H_3} \qquad \blacksquare$$

Table 2.4: Block diagram transformations.

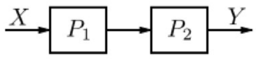
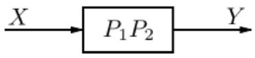
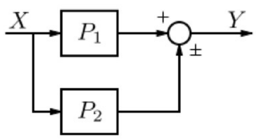
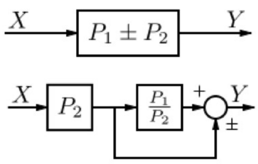
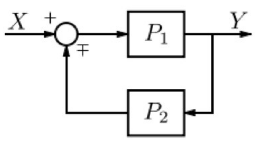
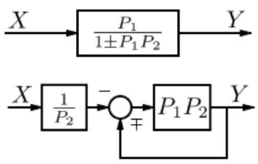
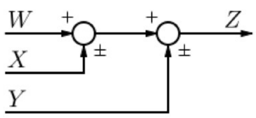
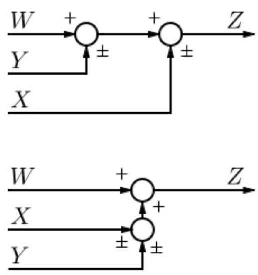
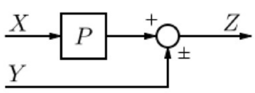
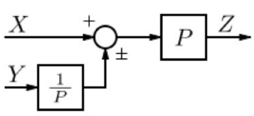
	Transformation	*Equation*	*Block diagram*	*Equivalent block diagram*
1.	Cascaded blocks	$Y = (P_1P_2)X$		
2.	Combining blocks in parallel	$Y = P_1X \pm P_2X$		
3.	Removing a block from a forward loop	$Y = P_1X \pm P_2X$		
4.	Eliminating feedback loop	$Y = P_1(X \mp P_2Y)$		
5.	Removing a block from a feedback loop	$Y = P_1(X \mp P_2Y)$		
6.	Rearranging summing junctions	$Z = W \pm X \pm Y$		
7.	Moving a summing junction in front of a block	$Z = PX \pm Y$		

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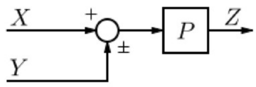
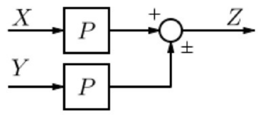
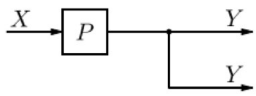
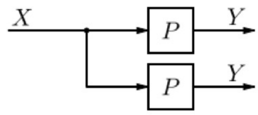
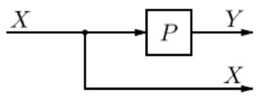
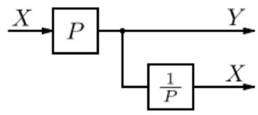
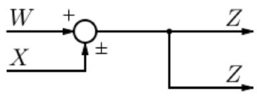
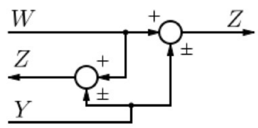
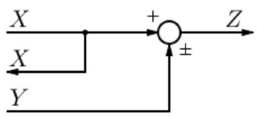
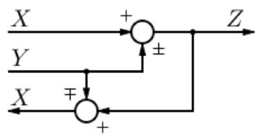
Table 2.4: .... continued.

8.	Moving a summing junction beyond a block	$Z = P(X \pm Y)$		
9.	Moving a takeoff point in front of a block	$Y = PX$		
10.	Moving a takeoff point beyond a block	$Y = PX$		
11.	Moving a takeoff point in front of a summing junction	$Z = X \pm Y$		
12.	Moving a takeoff point beyond a summing junction	$Z = X \pm Y$		

Systems with multiple inputs

In feedback control systems, we often encounter multiple inputs. For a linear system, we can apply the *principle of superposition* to solve this type of problems, i.e. to treat each input one at a time while setting all other inputs to zeros, and then algebraically add all the outputs as follows:

1. Set all inputs except one equal to zero.
2. Transform the block diagram to solvable form.
3. Find the output response due to the chosen input action alone.
4. Repeat steps 1 to 3 for each of the remaining inputs.
5. Algebraically summ all the output responses found in steps 1 to 5.

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Example 2.16

Find the complete output for the system shown in Figure 2.19 when both inputs act simultaneously.

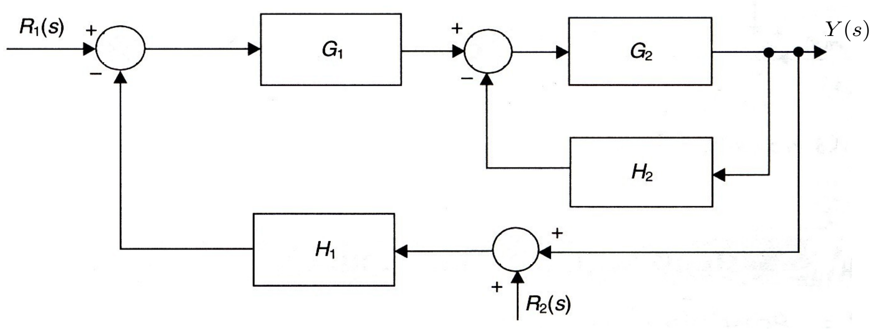

Figure 2.19: System with multiple inputs.

■ **Solution** The block diagram shown in Figure 2.19 can be reduced and simplified to the form given in Figure 2.20.

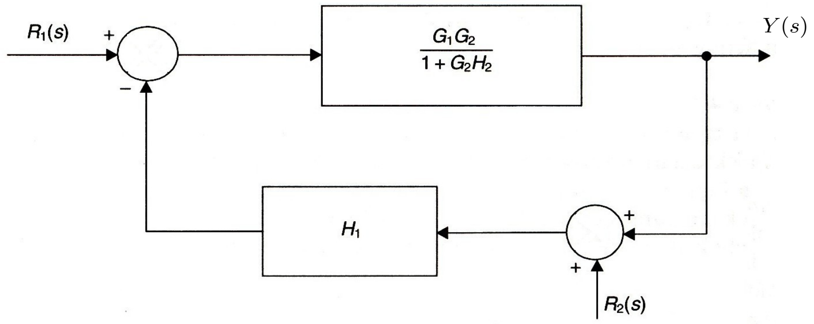

Figure 2.20: Reduced and simplified block diagram.

Putting $R_2(s) = 0$ and replacing the summing point by $+1$ gives the block diagram shown in Figure 2.21. In Figure 2.21 nothe that $Y_1(s)$ is response to $R_1(s)$ acting alone.

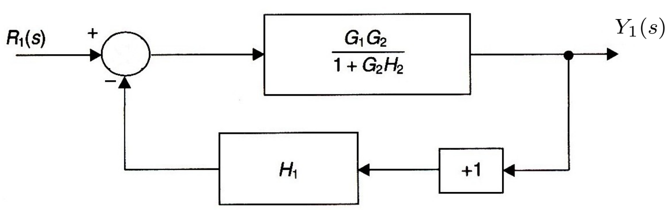

Figure 2.21: Block diagram for $R_1(s)$ acting alone.

The closed-loop transfer function is therefore

$$\frac{Y_1(s)}{R_1(s)} = \frac{\frac{G_1G_2}{1+G_2H_2}}{1 + \frac{G_1G_2H_1}{1+G_2H_2}}$$

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or

$$Y_1(s) = \frac{G_1G_2R_1}{1 + G_2H_2 + G_1G_2H_1}$$

Now if $R_1(s) = 0$ and the summing point is replaced by -1, then the response $Y_2(s)$ to input $R_2(s)$ acting alone is given by Figure 2.22. The choice as to whether the summing point is replaced by $+1$ or -1 depends upon the sign at the summing point.

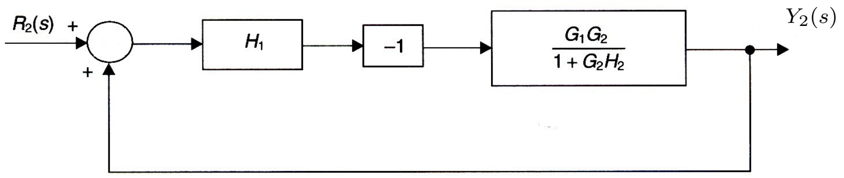

Figure 2.22: Block diagram for $R_2(s)$ acting alone.

Note that in Figure 2.22 there is a positive feedback loop. Hence the closed-loop transfer functionn relating $R_2(s)$ and $Y_2(s)$ is

$$\frac{Y_2(s)}{R_2(s)} = \frac{\frac{-G_1G_2H_1}{1+G_2H_2}}{1 - \left(\frac{-G_1G_2H_1}{1+G_2H_2}\right)}$$

or

$$Y_2(s) = \frac{-G_1G_2H_1R_2}{1 + G_2H_2 + G_1G_2H_1}$$

Using the principle of superposition, the complete response is given by

$$Y(s) = Y_1(s) + Y_2(s)$$

or

$$Y(s) = \frac{G_1G_2R_1 - G_1G_2H_1R_2}{1 + G_2H_2 + G_1G_2H_1} \qquad \blacksquare$$

2.7.2 Signal-flow graph

Block diagrams are adequate for the representation of the interrelationships of controlled and input variables. However, for a system with reasonably complex interrelationships, the block diagram reduction technique is cumbersome and often quite difficult to complete. An alternative method for determining the relationship between system variables has been developed by Mason and is based on a representation of the linear system by line segments called Signal-Flow Graph (SFG). The advantage of the SFG method is the availability of a flow graph gain formula, which provides the relation between system variables without requiring any reduction procedure or manipulation of the flow graph.

Basic elements of an SFG

A signal flow graph is a pictorial representation a set of simultaneous linear algebraic equations describing a system. When constructing a SFG, junction

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points or nodes are used to represent variables. The nodes are connected by line segments, called branches. A signal can transmit through a branch only in the direction of the arrow. As an example consider a linear system represented by a simple algebraic equation

$$y_2 = a_{12}y_1$$

where y_1 is the input, y_2 is the output, and a_{12} is the gain between the two variables. The SFG representation is shown in Figure 2.23

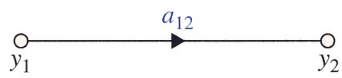

Figure 2.23: Signal-flow graph of $y_2 = a_{12}y_1$.

Example 2.17

Construct a SFG to the following set of algebraic equations:

$$\begin{aligned} y_2 &= a_{12}y_1 + a_{32}y_3 \\ y_3 &= a_{23}y_2 + a_{43}y_4 \\ y_4 &= a_{24}y_2 + a_{34}y_3 + a_{44}y_4 \\ y_5 &= a_{25}y_2 + a_{45}y_4 \end{aligned}$$

■ **Solution** Figure 2.24 shows a step-by-step construction of the signal-flow graph. The complete SFG is shown in Figure 2.25. ■

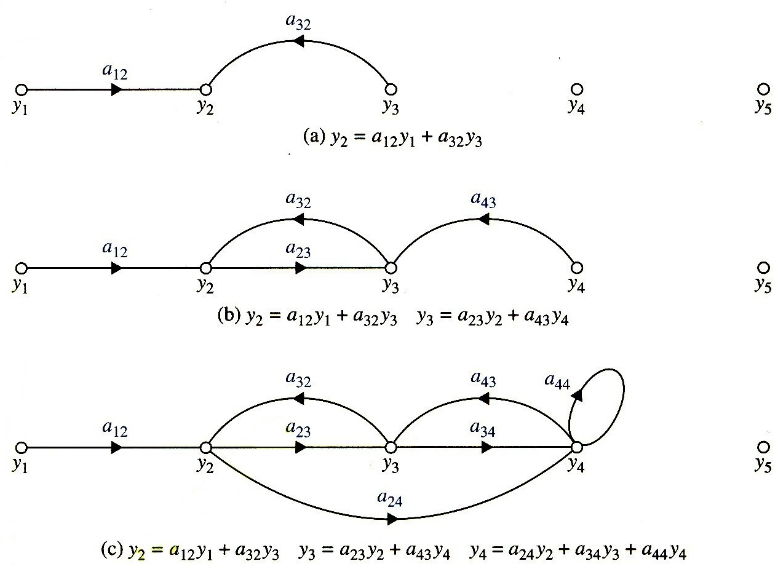

Figure 2.24: Step-by-step construction of the signal-flow graph.

BASIC PROPERTIES OF SFG

The important properties of the SFG are summarised as follows:

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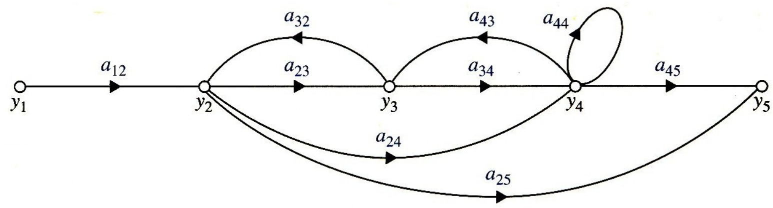

Figure 2.25: Complete signal-flow graph.

1. SFG applies only to linear systems.
2. The equations of which SFG is drawn must be algebraic equations in the form of cause and effect.
3. Nodes are used to represent variables. Normally, the nodes are arranged from left to right, from the input to the output, following succession of cause-and-effect relations through the system.
4. Signals travel along branches only in the direction described by the arrows of the branches.
5. The branch directing from y_k to y_j represents the dependence of y_j upon y_k but not the reverse.
6. A signal y_k traveling along a branch between y_k and y_j is multiplied by the gain of the branch a_{kj}, so that a signal $a_{kj}y_k$ is delivered at y_j.

Definitions of the SFG terms

Before proceeding further, we define some terms which we will need later:

- A **source** is a node which has outgoing branches only (Example: y_1 in Figure 2.25).
- A **sink** is a node which has only incoming branches (Example: y_5 in Figure 2.25).
- A **path** is a set of branches having the same sense of direction.
- A **forward path** originates from a source and terminates in a sink. No node may be encountered more than once. In the SFG of Figure 2.25, there are three forward paths between y_1 and y_5. One such path for example is from y_1 to y_2 to y_5 (through the branch with gain a_{25}).
- The **path gain** is the product of the coefficients associated with the branches of the path. For example the path gain for the path $y_1 - y_2 - y_5$ in Figure 2.25 is $a_{12}a_{25}$.
- A **feedback loop** is a path that begins and ends at the same node; additionally no node may be encountered more than once. For example there are four loops in the SFG of Figure 2.25. These are shown in Figure 2.26.
- The **loop gain** is the path gain of a feedback loop. For example, the loop gain of the loop $y_2 - y_4 - y_3 - y_2$ in Figure 2.26 is $a_{24}a_{43}a_{32}$.

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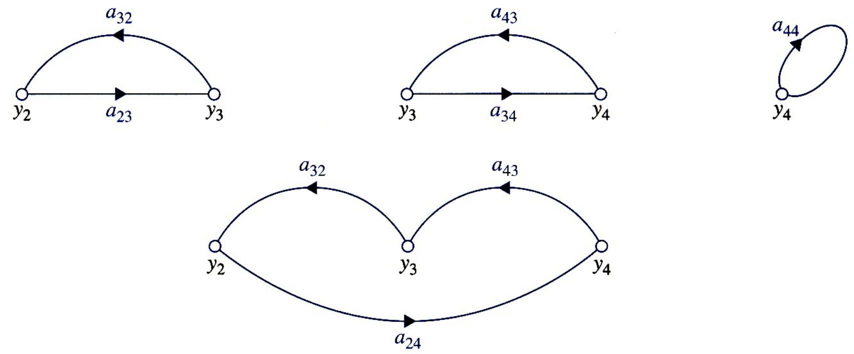

Figure 2.26: Four loops in the signal-flow graph of Figure 2.25.

Mason's Rule

Given an SFG or block diagram, the task of solving for the input-output relations by algebraic manipulation could be quite tedious. Fortunately, there is a general gain formula available that allows the determination of the input-output relations of an SFG by inspection.

Mason's states that the input-output transfer function associated with a signal-flow graph is given by

$$G = \frac{\sum_k P_k \Delta_k}{\Delta} \tag{2.11}$$

where

$$\Delta = 1 - \sum L_1 + \sum L_2 - \sum L_3 + \cdots + (-1)^m \sum L_m$$

and

$P_k =$ gain of the k^{th} forward path

$L_1 =$ gain of each closed loop in the graph

$L_2 =$ product of loop gains of any two nontouching loops (loops are called nontouching if they have no node in common)

$\vdots$

$L_m =$ product of loop gains of any m nontouching loops

$\Delta_k =$ the value of Δ remaining with the loops touching the path P_k are removed

Procedures to solve SFG by using Mason's rule:

1. Identify the no. of forward paths and determine the forward-path gains.
2. Identify the no. of loops and determine the loop gains.
3. Identify the non-touching loops taken two at a time, three at a time and so on. Determine the product of the non-touching loop gains.

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4. Determine Δ and Δ_k.
5. Substitute all of the above information into the Mason's gain formula.

Example 2.18 Determine the closed-loop transfer function $Y(s)/R(s)$ of the SFG shown in Figure 2.27.

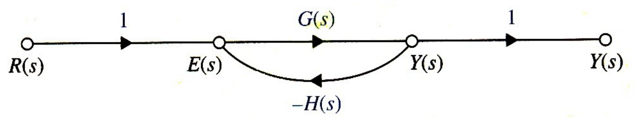

Figure 2.27: Signal-flow graph for example 2.18.

■ **Solution**

1. *Forward path:* There is only one forward path between $R(s)$ and $Y(s)$, and the forward-path gain is
$$P_1 = G(s)$$
2. *Closed loops:* There is only one loop; the loop gain
$$L_1 : \qquad -G(s)H(s)$$
3. *Non-touch loops:* There are no non-touching loops since there is only one loop.
4. Δ and Δ_1: The forward path is in touch with the only loop. Thus, $\Delta_1 = 1$, and
$$\Delta = 1 - L_1 = 1 + G(s)H(s)$$
5. Using (2.11), the closed-loop transfer function is written as
$$\frac{P_1\Delta_1}{\Delta} = \frac{G(s)}{1 + G(s)H(s)} \qquad ■$$

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Example 2.19

Determine the gain between y_1 and y_5 using the gain formula for the SFG shown in Figure 2.25.

■ **Solution**

1. *Forward path*: There are three forward paths between y_1 and y_5 and forward-path gains are

$$\begin{array}{lll} P_1 = a_{12}a_{23}a_{34}a_{45} & \text{Forward path:} & y_1 - y_2 - y_3 - y_4 - y_5 \\ P_2 = a_{12}a_{25} & \text{Forward path:} & y_1 - y_2 - y_5 \\ P_3 = a_{12}a_{24}a_{45} & \text{Forward path:} & y_1 - y_2 - y_4 - y_5 \end{array}$$

2. *Closed loops:* The four loops of the SFG are shown in Figure 2.26. The loop gains are

$$L_1: \quad a_{23}a_{32} \quad a_{34}a_{43} \quad a_{24}a_{43}a_{32} \quad a_{44}$$

hence

$$\sum L_1 = a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{43}a_{32} + a_{44}$$

3. *Non-touch loops:* There is only one pair of non-touching loops; that is, the two loops

$$y_2 - y_3 - y_2 \quad \text{and} \quad y_4 - y_4$$

Thus the product of the gains of the two non-touching loops is

$$L_2: \quad a_{23}a_{32}a_{44}$$

4. Δ and Δ_k: $\Delta = 1 - \sum L_1 + \sum L_2$ hence

$$\Delta = 1 - (a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{43}a_{32} + a_{44}) + a_{23}a_{32}a_{44}$$

All the loops are in touch with forward paths P_1 and P_3. Thus, $\Delta_1 = \Delta_3 = 1$. Two of the loops are not in touch with forward path P_2. These loops are: $y_3 - y_4 - y_3$ and $y_4 - y_4$. Thus,

$$\Delta_2 = 1 - a_{34}a_{43} - a_{44}$$

5. Using (2.11), the closed-loop transfer function is written as

$$\begin{aligned} G &= \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta} \\ &= \frac{a_{12}a_{23}a_{34}a_{45} + (a_{12}a_{25})(1 - a_{34}a_{43} - a_{44}) + a_{12}a_{24}a_{45}}{1 - (a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{43}a_{32} + a_{44}) + a_{23}a_{32}a_{44}} \end{aligned}$$ ■

Application of Mason's rule between a source and a non-sink node

In the previous example, Example 2.19, we basically determined the transfer function between y_1(source node) and y_5(sink node). Often, it is of interest to find the relation between a source and a non-sink node. For example, in the SFG of Figure 2.25, it may be of interest to find the relation y_2/y_1, which represents dependence of y_2 upon y_1; noting that y_2 is not a sink node. To make y_2 a sink

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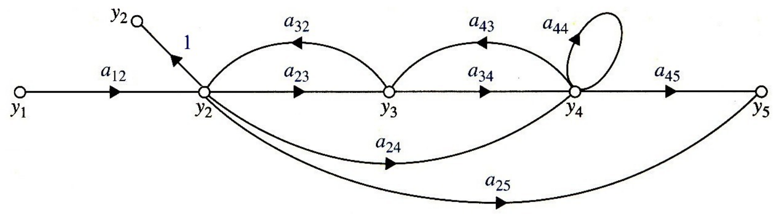

Figure 2.28: Modification of signal-flow graph so that y_2 satisfy the condition as sink node.

node, we simply connect a branch with unity gain from the existing node y_2 to a new node also designated as y_2, as shown in Figure 2.28.

Example 2.20 Determine the gain between y_1 and y_2 using the gain formula for the SFG shown in Figure 2.28.

■ **Solution**

1. *Forward path*: There is only one forward path between y_1 and y_2 and forward-path gain is
$$P_1 = a_{12} \qquad \text{Forward path:} \qquad y_1 - y_2$$
2. *Closed loops:* The four loops of the SFG are shown in Figure 2.26. The loop gains are
$$L_1: \qquad a_{23}a_{32} \qquad a_{34}a_{43} \qquad a_{24}a_{43}a_{32} \qquad a_{44}$$
hence
$$\sum L_1 = a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{43}a_{32} + a_{44}$$
3. *Non-touch loops:* There is only one pair of nontouching loops; that is, the two loops
$$y_2 - y_3 - y_2 \qquad \text{and} \qquad y_4 - y_4$$
Thus the product of the gains of the two nontouching loops is
$$L_2: \qquad a_{23}a_{32}a_{44}$$
4. Δ and Δ_k: $\Delta = 1 - (a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{43}a_{32} + a_{44}) + a_{23}a_{32}a_{44}$

   Two of the loops are not in touch with forward path P_1. These loops are: $y_3 - y_4 - y_3$ and $y_4 - y_4$. Thus,
$$\Delta_1 = 1 - a_{34}a_{43} - a_{44}$$
5. Using (2.11), the transfer function between y_1 and y_2 is written as
$$\begin{aligned} G &= \frac{P_1\Delta_1}{\Delta} \\ &= \frac{a_{12}(1 - a_{34}a_{43} - a_{44})}{1 - (a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{43}a_{32} + a_{44}) + a_{23}a_{32}a_{44}} \end{aligned}$$
Note that Δ is the same as in Example 2.19 regardless of which sink node is chosen. ■

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Application of Mason's rule between a non-source and sink node

We have seen earlier how to determine the gain between a source and a non-sink node. Another situation of interest is find the relation between a non-source and a sink node. For example, in the SFG of Figure 2.29, it may be of interest to find the relation y_7/y_2, which represents the dependence of y_7 upon y_2, the latter is not a source node.

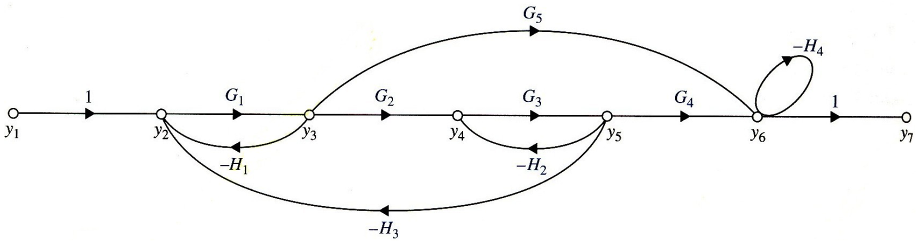

Figure 2.29: Signal-flow graph for Example 2.21.

We can show that by including a source node (y_1 in this case), we may write y_7/y_2 as

$$\frac{y_7}{y_2} = \frac{y_7/y_1}{y_2/y_1} = \frac{\left.\frac{\sum_k P_k \Delta_k}{\Delta}\right|_{\text{from } y_1 \text{ to } y_7}}{\left.\frac{\sum_k P_k \Delta_k}{\Delta}\right|_{\text{from } y_1 \text{ to } y_2}}$$

Since Δ is independent of the sources and sinks, the last equation is written

$$\frac{y_7}{y_2} = \frac{y_7/y_1}{y_2/y_1} = \frac{\left.\sum_k P_k \Delta_k\right|_{\text{from } y_1 \text{ to } y_7}}{\left.\sum_k P_k \Delta_k\right|_{\text{from } y_1 \text{ to } y_2}}$$

Note that Δ does not appear in the last equation. However, you must evaluate it to be able to find Δ_k.

Example 2.21

Determine the gain between y_7 and y_2 for the SFG shown in Figure 2.29.

■ **Solution** We start by determining y_2/y_1:

1. *Forward path*: There is only one forward path between y_1 and y_2 and forward-path gain is

$$P_1 = 1 \qquad \text{Forward path:} \qquad y_1 - y_2$$

2. *Closed loops:* There are four loops

$$y_2 - y_3 - y_2 \qquad y_4 - y_5 - y_4 \qquad y_2 - y_3 - y_4 - y_5 - y_2 \qquad y_4$$

with gains

$$L_1 : \qquad -G_1H_1 \qquad -G_3H_2 \qquad -G_1G_2G_3H_3 \qquad -H_4$$

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Hence,

$$\sum L_1 = -(G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + H_4)$$

3. *Non-touch loops*:
   - Product of loop gains of any two nontouching loops (there are four possible combinations), thus:

     $$L_2: \qquad G_1G_3H_1H_2 \qquad G_1H_1H_4 \qquad G_3H_2H_4 \qquad G_1G_2G_3H_3H_4$$

     and

     $$\sum L2 = G_1G_3H_1H_2 + G_1H_1H_4 + G_3H_2H_4 + G_1G_2G_3H_3H_4$$

   - Product of loop gains of any three nontouching loops:

     $$L_3: \qquad -G_1G_3H_1H_2H_4$$

4. Δ and Δ_k: $\Delta = 1 - \sum L_1 + \sum L_2 - \sum L_3$, therefore,

   $$\begin{aligned}\Delta = 1 &+ G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + H_4 + G_1G_3H_1H_2 \\ &+ G_1H_1H_4 + G_3H_2H_4 + G_1G_2G_3H_3H_4 + G_1G_3H_1H_2H_4\end{aligned}$$

   Two of the loops are not in touch with forward path P_1. These loops are: $y_4 - y_5 - y_4$ and $y_4 - y_4$. Thus,

   $$\Delta_1 = 1 + G_3H_2 + H_4 + G_3H_2H_4$$

5. Using (2.11), the gain between y_1 and y_2 is written as

   $$\frac{y_2}{y_1} = \frac{P_1\Delta_1}{\Delta} = \frac{1 + G_3H_2 + H_4 + G_3H_2H_4}{\Delta}$$

In a similar fashion we now determine the gain between y_1 and y_7:

1. *Forward path*: There are two forward path between y_1 and y_7, the forward-paths gains are

   $$\begin{aligned}P_1 &= G_1G_2G_3G_4 \qquad \text{Forward path:} \qquad y_1 - y_7 \\ P_2 &= G_1G_5 \qquad \text{Forward path:} \qquad y_1 - y_2 - y_3 - y_6 - y_7\end{aligned}$$

2. *Closed loops:* As before.
3. *Non-touch loops*: As before
4. Δ and Δ_k: The forward path P_1 is in touch with all loops. Thus, $\Delta_1 = 1$. One loop is not in touch with forward path P_2, $y_4 - y_5 - y_4$, thus

   $$\Delta_2 = 1 + G_3H_2$$

5. Using (2.11), the gain between y_1 and y_7 is written as

   $$\frac{y_7}{y_1} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4 + G_1G_5(1 + G_3H_2)}{\Delta}$$

Finally, the gain between y_2 and y_7 is

$$\frac{y_7}{y_2} = \frac{y_7/y_1}{y_2/y_1} = \frac{G_1G_2G_3G_4 + G_1G_5(1 + G_3H_2)}{1 + G_3H_2 + H_4 + G_3H_2H_4} \qquad \blacksquare$$

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2.7.3 Conversion from block diagram to SFG

An equivalent SFG for a block diagram can be drawn by performing the following steps:

1. Identify the input/output signals, summing junctions & pickoff points→they are replaced with nodes.
2. Interconnect the nodes & indicate the directions of signal flow by using arrows.
3. Identify the blocks → they are replaced with branches. For each negative sum, a negative sign is included with the branch.
4. Label the input/output signals and the branches accordingly.
5. Add unity branches as needed for clarity or to make connections.
6. Simplify the SFG → eliminate redundant nodes/branches (only if the node is connected to branches of a single flow in & a single flow out with unity gain).

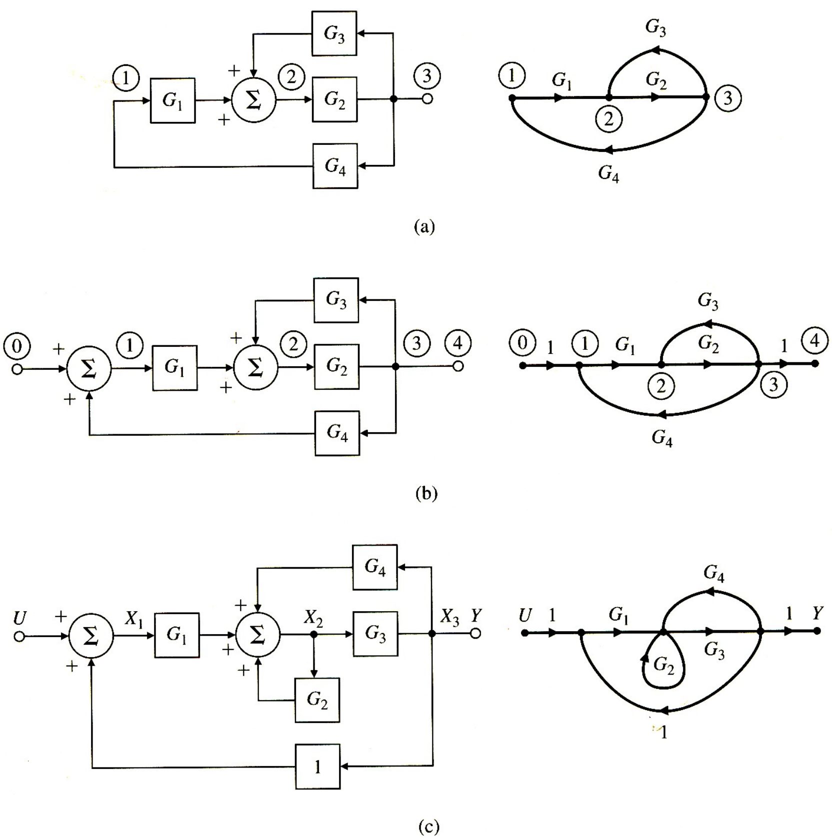

Figure 2.30: Block diagrams and corresponding signal-flow graphs.

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Example 2.22 Convert the block diagram in Figure 2.31 to a signal flow graph and determine the transfer function using Mason's gain formula.

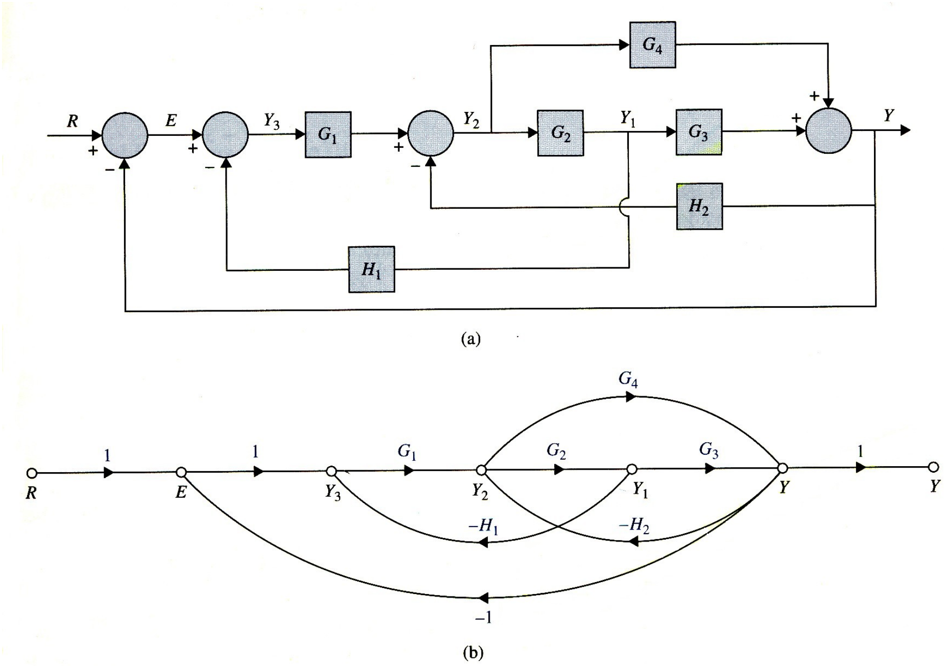

Figure 2.31: (a) Block diagram a control system. (b) Equivalent signal-flow graph.

■ Solution

1. *Forward path*:

$$P_1 = G_1G_2G_3$$
$$P_2 = G_1G_4$$

2. *Closed loops:*

L_1 : $-G_1G_2H_1$ $-G_2G_3H_2$ $-G_1G_2G_3$ $-G_1G_4$ $-G_4H_2$

3. *Non-touch loops*: There are no nontouching loops.

4. Δ and Δ_k: $\Delta = 1 - \sum L_1 + \sum L_2 - \sum L_3$, therefore,

$$\Delta = 1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 + G_1G_4 + G_4H_2$$

All the loops are in touch with P_1 and P_2, thus $\Delta_1 = \Delta_2 = 1$.

5. Using (2.11), the transfer function between $Y(s)$ and $R(s)$ is written as

$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_1G_4}{\Delta} \quad \blacksquare$$

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2.7.4 Construction of block diagrams Examples

Example 2.23

Construct a block diagram for the mechanical system described by the following set of equations

$$X_1(s) = \frac{B_2 s + K}{m_1 s^2 + (B_1 + B_2)s + k} X_3(s) + \frac{F(s)}{m_1 s^2 + (B_1 + B_2)s + k} \tag{2.12}$$

$$X_2(s) = \frac{B_3}{m_2 s + B_3 + B_4} X_3(s) \tag{2.13}$$

$$X_3(s) = \frac{B_2 s + K}{(B_2 + B_3)s + k} X_1(s) + \frac{B_3 s}{(B_2 + B_3)s + k} X_2(s) \tag{2.14}$$

■ **Solution** Each dynamic equation represents a subsystem. Its block diagram is constructed by a simple principle: treat the right hand side signals as the input and the left hand side as the output. Equation (2.12) can be represented as

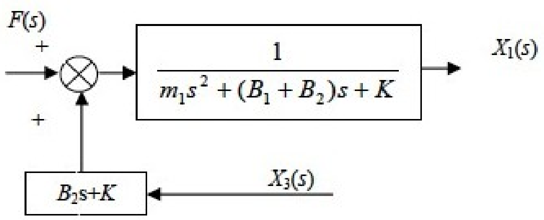

Figure 2.32: Block diagram of Equation (2.12).

Equation (2.13) can be represented as

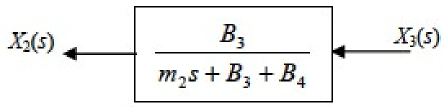

Figure 2.33: Block diagram of Equation (2.13).

and Equation (2.14)

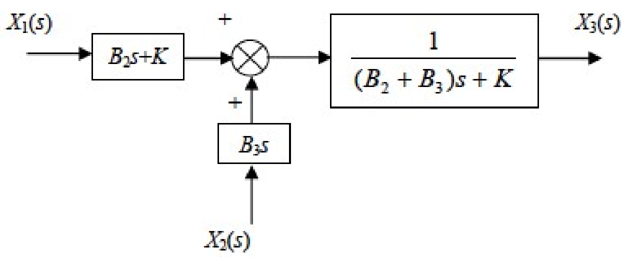

Figure 2.34: Block diagram of Equation (2.14).

After construction of block diagrams for individual equations, we connect these

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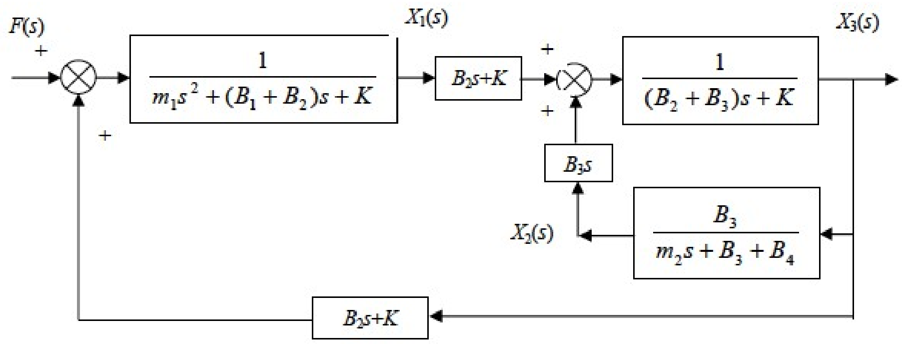

Figure 2.35: Block diagram of the system in Example 2.23.

block diagrams to form a block diagram for the entire system:

Example 2.24

Construct a block diagram for a system described by the following set of equations

$$P_d(s) = \frac{1}{CR_d s + 1} P_c(s) \tag{2.15}$$

$$P_i(s) = \frac{1}{CR_i s + 1} P_c(s) \tag{2.16}$$

$$P_c(s) = KX(s) \tag{2.17}$$

$$X(s) = \frac{b}{a+b} E(s) + \frac{a}{a+b} Y(s) \tag{2.18}$$

$$Y(s) = \frac{A}{K_s} \left[P_i(s) - P_d(s) \right] \tag{2.19}$$

■ **Solution** Each dynamic equation represents a subsystem. Its block diagram is constructed by a simple principle: treat the right hand side signals as the input and the left hand side as the output.

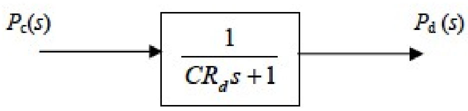

Figure 2.36: Block diagram of Equation (2.15).

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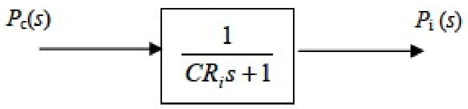

Figure 2.37: Block diagram of Equation (2.16).

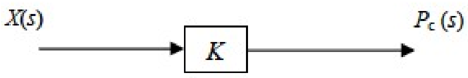

Figure 2.38: Block diagram of Equation (2.17).

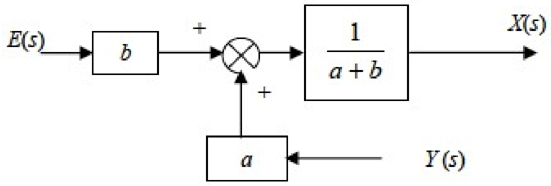

Figure 2.39: Block diagram of Equation (2.18).

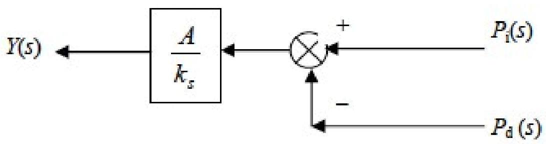

Figure 2.40: Block diagram of Equation (2.19).

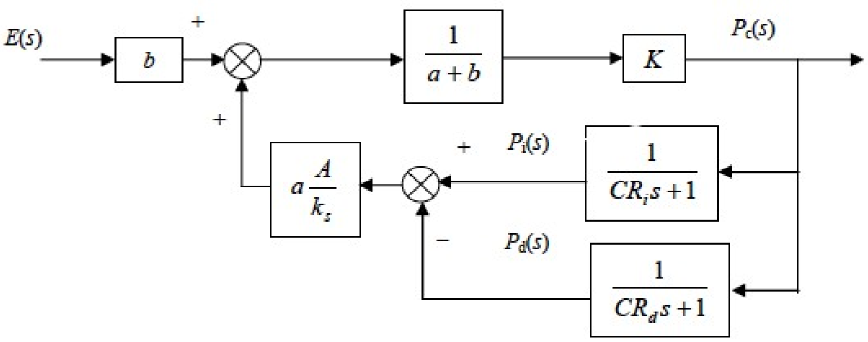

Figure 2.41: Block diagram of the system in Example 2.24.