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Chapter 3

Time-Domain Analysis of Control Systems

3.1 Introduction

This chapter refers to the time-domain analysis of linear time-invariant control systems. The problem of time-domain analysis may be briefly stated as follows: given the system (i.e., given a specific description of the system) and its input, determine the time-domain behavior of the output of the system.

In the analysis problem, we will use selected input signals to test the response of control systems. This response will be characterized by a selected set of response measures. The basic motivation for system analysis is that one can predict (theoretically) the system's behavior.

3.2 System Time Responses

The manner in which a dynamic system responds to an input, expressed as a function of time, is called the time response. It is possible to compute the time response of a system if the following is known:

- the nature of the input, expressed as a function of time
- the mathematical model of the system.

The time response of any control system has two components:

(a) Transient response: Is that particular part of the response of the system which tends to zero as time increases. It is a function only of the system dynamics, and is independent of the input signal.

(b) Steady-state response: Is that particular part of the response of the system that remains after the transient component has reached zero. It is a function of both the system dynamics and the input signal.

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The total response of the system is always the sum of the transient and steady-state components. In the design problem, specifications are usually given in terms of the transient and steady-state performance, and controllers are designed so that the specifications are all met by the design system.

For control system design, we need a basis of comparison of performance of different designs. One way of setting up this basis is to specify particular test signals and compare the response of different designs to these signals.

3.3 Common test Signals

All the following signals are defined for $t \geq 0$.

3.3.1 The unit impulse function

The unit impulse function is based on a rectangular function $\delta_\epsilon(t)$ such that

$$\delta_\epsilon(t) = \frac{1}{\epsilon}, \qquad 0 \leq t \leq \epsilon, \qquad \epsilon > 0$$

As $\epsilon \to 0$, $\delta_\epsilon(t)$ approaches the unit impulse $\delta(t)$. The major properties of this function are

1. $\int_0^\infty \delta(t)dt = 1$
2. $\int_0^\infty g(t)\delta(t-a)dt = g(a)$
3. $\mathcal{L}[\delta(t)] = 1$

It is very useful for modeling shock inputs.

3.3.2 The unit step function

The unit step function is defined as:

$$r(t) = 1, \quad t \geq 0 \qquad \Longrightarrow \mathcal{L}[r(t)] = \frac{1}{s}$$

and is useful for modeling sudden disturbances.

3.3.3 The unit ramp function

The unit ramp function is defined as:

$$r(t) = t, \quad t \geq 0 \qquad \Longrightarrow \mathcal{L}[r(t)] = \frac{1}{s^2}$$

It is useful for modeling gradually changing inputs.

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3.3.4 The sinusoidal functions

The sinusoidal functions

$$\cos \omega t = \text{Re } e^{j\omega t} \implies \mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$
$$\sin \omega t = \text{Im } e^{j\omega t} \implies \mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

are very important in frequancy response techniques.

3.4 Response of first order systems

3.4.1 Standard form

Consider a first-order differential equation

$$a\frac{dy(t)}{dt} + by(t) = cr(t)$$

Take Laplace transoform with zero initial conditions

$$asY(s) + bY(s) = c\,R(s)$$
$$(as + b)Y(s) = c\,R(s)$$

The first order transfer function is

$$G(s) = \frac{Y(s)}{R(s)} = \frac{c}{as + b}$$

To obtain the standard form, divide by b

$$G(s) = \frac{c/b}{1 + (a/b)s}$$

which is written as

$$G(s) = \frac{K}{1 + \tau s}$$

3.4.2 Step response

The response of the control system to the unit-step input is called the unit-step response. When $r(t)$ is a unit step

$$Y(s) = \frac{K}{s(\tau s + 1)} = K\left(\frac{1}{s} - \frac{1}{s + 1/\tau}\right)$$

$$\implies y(t) = K(1 - e^{-t/\tau}) = K - Ke^{-t/\tau} \tag{3.1}$$

The first term is called the steady-state response and K is called the steady-state value. The second term is called the transient response. If $\tau > 0$ the transient response tends to 0 as $t \to \infty$. The step response of a first-order system as given in (3.1) is shown in Figure 3.1. Note that the exponentially decaying term has an initial slope of K/τ; that is,

$$\frac{d}{dt}\left(K - Ke^{-t/\tau}\right)_{t=0} = \frac{K}{\tau}e^{-t/\tau}\bigg|_{t=0} = \frac{K}{\tau}$$

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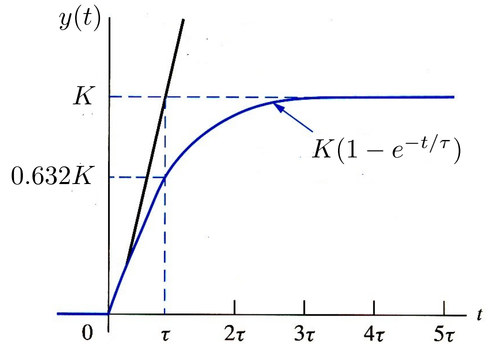

Figure 3.1: Step response of first -order systems.

Mathematically, the exponential term does not decay to zero in a finite length of time. However, if the term continued to decay at its initial rate, it would reach a value of zero in τ seconds. The parameter τ is called the time constant and has the units of seconds. The exponential function decays to about 2% of its initial value within 4 time constants. The output $y(t)$ reaches about 63% of its final vaue when $t = \tau$.

Example 3.1 An example of a first order system is provided by the following RC circuit.

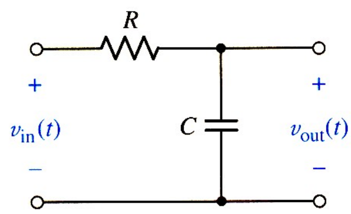

Figure 3.2: A simple resistor-capacitor circuit.

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{1}{1 + sRC} \qquad (K = 1, \quad \tau = RC) \quad \blacksquare$$

3.4.3 Steady-state response

The concept of finding the steady-state response to a unit step for a system of any order is now developed. Suppose that

$$Y(s) = G(s)R(s)$$

where $G(s)$ is a given transfer function. From Section 2.2.3, the final-value theorem of the Laplace transform is

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)R(s)$$

provided that the limit

$$\lim_{t\to\infty} y(t)$$

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exists, i.e., $y(t)$ has a final value. For the case that the input is a unit step, $R(s)$ is equal to $1/s$ and

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sG(s)\frac{1}{s} = \lim_{s\to 0} G(s) \\ &= G(0)\end{aligned} \tag{3.2}$$

$G(0)$ is often called the dc gain of the system and is defined as the ratio of the output of a system to a constant input after all transients has decayed. Care must be taken to apply the Final value Theorem only to stable systems (i.e., $y(t)$ is bounded) and with at most a single pole at $s = 0$.

Example 3.2

Find the dc gain of the system whose transfer function is

$$G(s) = \frac{3(s+2)}{(s^2+2s+10)}$$

■ **Solution** Applying (3.2), we get

$$\text{dc gain} = G(s)\Big|_{s=0} = \frac{(3)(2)}{(10)} = 0.6 \quad ■$$

3.5 Response of second order systems

3.5.1 Standard form

Consider a second-order differential equation

$$a\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = er(t)$$

Take Laplace transforms, with zero initial conditions

$$\begin{aligned}as^2Y(s) + bsY(s) + CY(s) &= eR(s) \\ (as^2 + bs + c)Y(s) &= eR(s)\end{aligned}$$

The transfer function is

$$G(s) = \frac{Y(s)}{R(s)} = \frac{e}{as^2+bs+c}$$

To obtain the standard form, divide by c

$$G(s) = \frac{e/c}{(a/c)s^2 + (b/c)s + 1}$$

which is written as

$$G(s) = \frac{K}{(1/\omega_n^2)s^2 + (2\zeta/\omega_n)s + 1}$$

and with the s^2 coefficient normalized to unity

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where

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ζ : damping ratio

ω_n : undamped natural frequency

K : dc gain

Since K only determines the dc magnitude of the response, we will set $K = 1$ and so

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{3.3}$$

where

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

are the roots of the characteristic equation and are the poles of $G(s)$.

3.5.2 Pole locations

All system characteristics of the standard second-order system are functions of only ζ and ω_n, since ζ and ω_n are the only parameters that appear in the transfer function (3.3). There are four cases of practical interest:

Case 1: $\zeta > 1$ (Overdamped system)

When $\zeta > 1$ the roots $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ are real, negative and unequal.

$\zeta > 1$

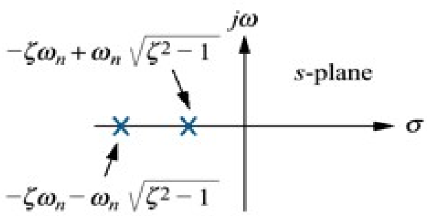

Figure 3.3: Pole locations in the s-plane when $\zeta > 1$.

Case 2: $\zeta = 1$ (Critically damped system)

When $\zeta = 1$ the roots $s_1, s_2 = -\zeta\omega_n$ are real, negative and equal.

$\zeta = 1$

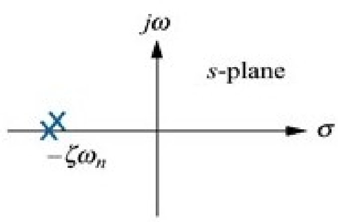

Figure 3.4: Pole locations in the s-plane when $\zeta = 1$.

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Case 3: $0 \leq \zeta < 1$ (Underdamped system)

When $0 \leq \zeta < 1$ the roots

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

are complex conjugate and have negative real parts. The real parts are zero if $\zeta = 0$ and the system is called undamped.

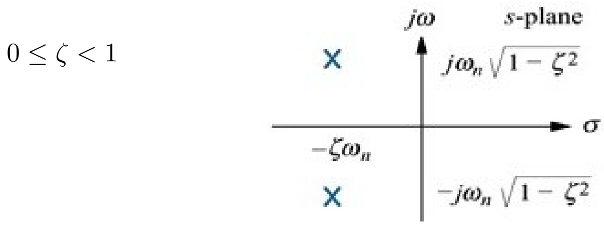

Figure 3.5: Pole locations in the s-plane when $0 \leq \zeta < 1$.

Case 4: $\zeta < 0$ (Unstable system)

When $\zeta < 0$ the roots s_1, s_2 have positive real parts and will be covered later.

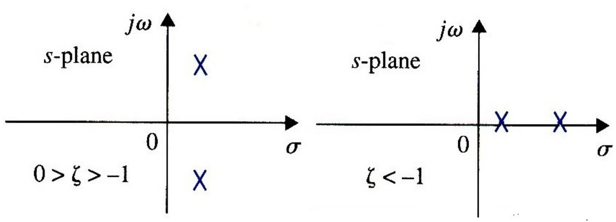

Figure 3.6: Pole locations in the s-plane when $\zeta < 0$.

Figure 3.7 summarizes all four cases.

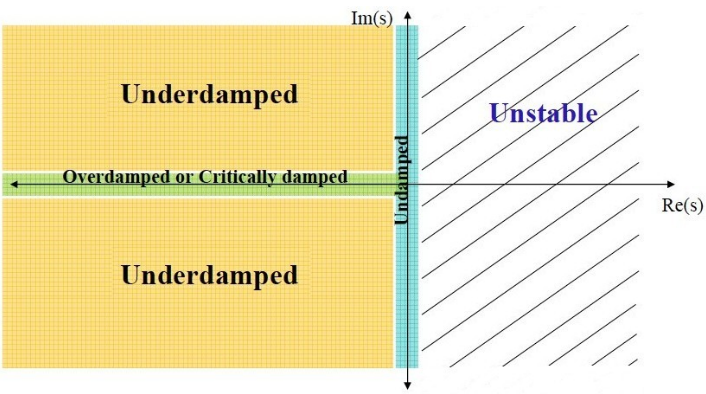

Figure 3.7: Pole locations in the s-plane and the corresponding transient response type.

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3.5.3 Step response

Consider the second-order system described by the transfer function given in (3.3), the step response is

$$Y(s) = G(s)R(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Expanding in partial fraction

$$Y(s) = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Using the cover-up method, we find

$$K_1 = sY(s)\Big|_{s=0} = 1$$

Equating the numerators,

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) + K_2 s^2 + K_3 s = \omega_n^2$$

After equating the powers of s on the two sides of the above equation, we find that $K_2 = -1$ and $K_3 = -2\zeta\omega_n$, hence

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Completing the square

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2 - \zeta^2\omega_n^2} \\ &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \left(\omega_n\sqrt{1 - \zeta^2}\right)^2} \end{aligned}$$

and writing it in the standard forms in Table 2.1 we have

$$Y(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] \tag{3.4}$$

where

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

is called the damped natural frequency. Taking the inverse Laplace transform

$$\begin{aligned} y(t) &= 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1 - \zeta^2}}\left[e^{-\zeta\omega_n t}\sin\omega_d t\right] \\ &= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}}\sin\omega_d t\right) \end{aligned} \tag{3.5}$$

Using the trigonometric identity, $\sin(a + b) = \sin a\cos b + \cos a\sin b$, (3.5) can be written as

$$y(t) = 1 - \frac{1}{\sqrt{1 - \zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t + \alpha) \tag{3.6}$$

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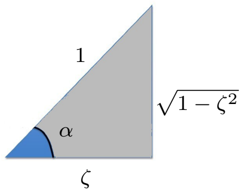
where

$$\alpha = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \tag{3.7}$$



is as shown (note that $\cos\alpha = \zeta$ and $\sin\alpha = \sqrt{1-\zeta^2}$). The step responses for a second-order system are shown in Figure 3.8 for several values of ζ as a function of $\omega_n t$.

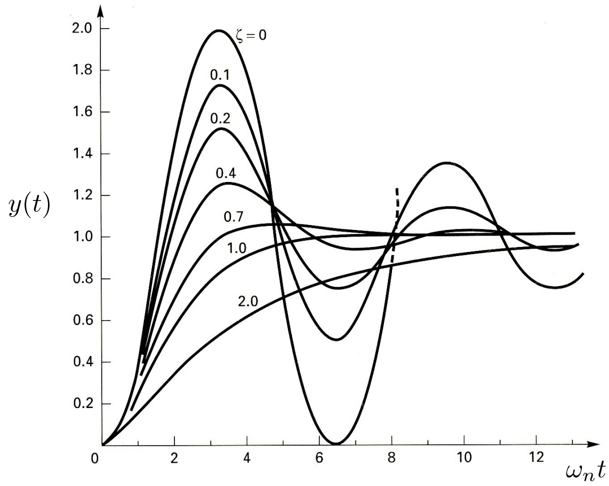


Figure 3.8: Step response for second-order system.

With reference to Figure 3.8:

- If $\zeta = 0$, from (3.5), noting that $\omega_d = \omega_n$

$$y(t) = 1 - \cos\omega_n t$$

  the frequency of the sinusoid is ω_n, called the undamped natural frequency. The step input will cause the system to oscillate continuously at ω_n.

- If $0 < \zeta < 1$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the frequency of the sinusoid, hence, damped natural frequency. The response is a damped sinusoid, and is called damped (or underdamped) system. The time constant of the exponential envelope is $\tau = \frac{1}{\zeta\omega_n}$. As ζ increases from 0 to 1, the response becomes less oscillatory, and hence the name damping ratio.

- If $\zeta \geq 1$, the oscillations have ceased and hence the system is called an overdamped system when $\zeta > 1$ and critically damped when $\zeta = 1$. For the case $\zeta = 1$ it is not clear from (3.5) how the oscillations in $y(t)$ have ceased. However, before taking the inverse Laplace to $Y(s)$ in (3.4) let $\zeta = 1$ to obtain

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s+\omega_n}{(s+\omega_n)^2+0} - \frac{\omega_n}{\omega_d}\left[\frac{\omega_d}{(s+\omega_n)^2+0}\right] \\ &= \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2} \end{aligned}$$

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Taking inverse Laplace transform

$$\begin{aligned} y(t) &= 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t} \\ &= 1 - e^{-\omega_n t}(1 + \omega_n t) \end{aligned}$$

which clearly shows that no sinusoidal term exist. For the overdamped case, we have two real poles at $-\zeta\omega_n \pm \omega_d$. The corresponding response is easily obtained from

$$Y(s) = \frac{1}{s} + \frac{K_2}{s + \zeta\omega_n + \omega_d} + \frac{K_3}{s + \zeta\omega_n - \omega_d}$$

as

$$y(t) = 1 + K_2 e^{-(\zeta\omega_n + \omega_d)t} + K_3 e^{-(\zeta\omega_n - \omega_d)t}$$

As an exercise find K_2 and K_3.

The effect of the characteristic equation roots on the damping of the second-order system is further illustrated by Figure 3.9

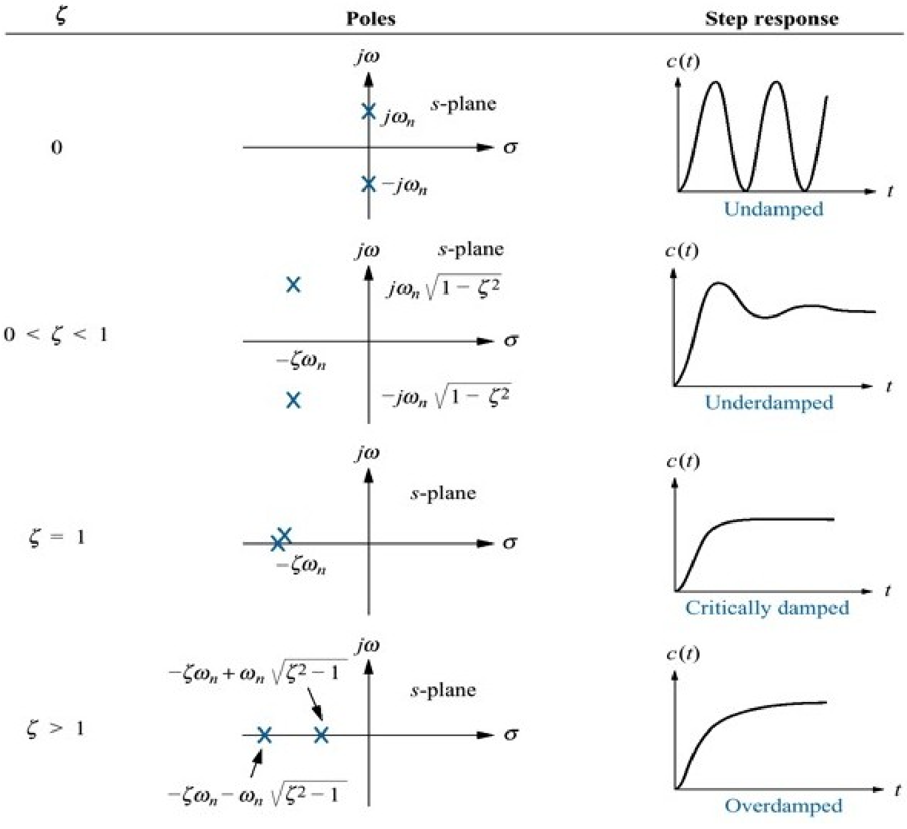

Figure 3.9: Step response comparison for various pole locations in the s-plane.

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3.6 Specifications of a second order system

In designing control systems, specifications must be developed that describes the characteristics the system should posses. Usually system design specifications or standard performance measures can be described in terms of the step response of a system as shown in Figure 3.10.

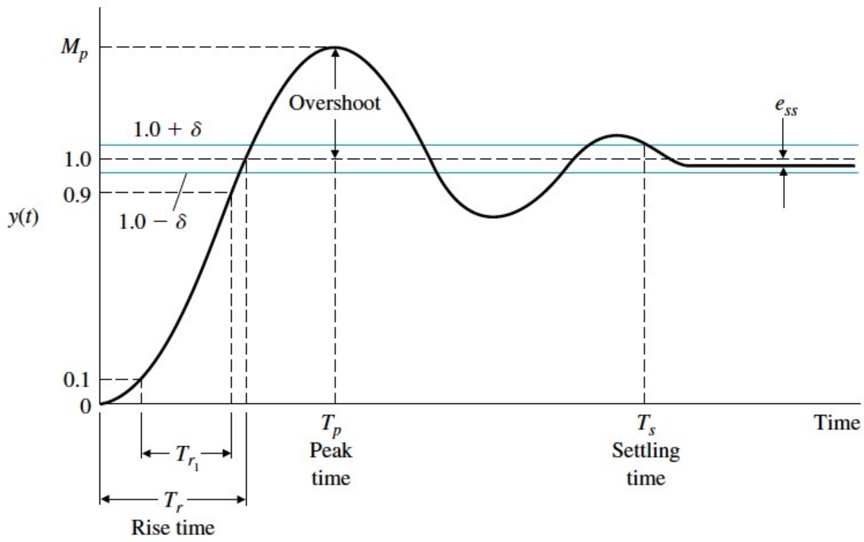

Figure 3.10: Step response of a control system.

For underdamped systems, we define the following specifications:

$$\begin{aligned} T_r &: \text{rise time (0\%-100\%)} \\ T_{r_1} &: \text{rise time (10\%-90\%)} \\ M_p &: \text{peak value} \\ T_p &: \text{time to first peak} \\ y_{ss} &: \text{steady state value} \\ \text{\% overshoot} &: \frac{M_p - y_{ss}}{y_{ss}} \times 100\% \\ T_s &: \text{settling time} \end{aligned}$$

The speed of the response is measured by the **rise time**, T_r, and the **peak time**, T_p. For underdamped systems with an overshoot, the $0 - 100\%$ rise time is a useful index. If the system is overdamped, then the peak time is not defined, and the $10 - 90\%$ rise time, T_{r_1}, is normally used.

The tracking properties, i.e., the similarity with which the actual response matches the step input is measured by the **percent overshoot** and **settling time**, T_s. The % overshoot is defined as

$$\text{\% overshoot} = \frac{M_p - y_{ss}}{y_{ss}} \times 100\% \tag{3.8}$$

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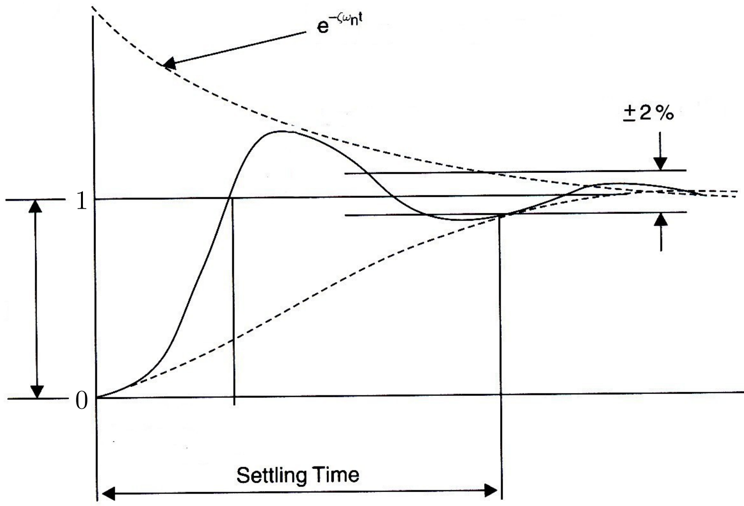

Figure 3.11: Step response of a control system.

where M_p is the peak value of the time response, and y_{ss} is the steady state value or the final value of the response.

The settling time, T_s, is defined as the time required for the system to settle within a certain percentage, δ, of the input amplitude. This band of $\pm\delta$ is shown in Figure 3.10. In other words it is the time required for the transient to decay to a small value so that $y(t)$ is almost in the steady state. For second order systems we usually determine the time, T_s, for which the response remains within 2% of the final value. From Figure 3.11 we can determine an approximate to T_s by computing the time when the decaying exponential $e^{-\zeta\omega_n t}$ reaches 2%:

$$e^{-\zeta\omega_n T_s} = 0.02$$

or

$$\zeta\omega_n T_s \approx 4$$

Therefore, we have

$$T_s \approx 4\tau = \frac{4}{\zeta\omega_n}$$

The steady-state error of the system may be measured on the step response of the system as shown in Figure 3.10. To obtain analytic expressions for T_p we first differentiate (3.6) to obtain

$$\dot{y}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\sqrt{1-\zeta^2}\,\omega_n t$$

and equating $\dot{y}(t) = 0$ gives

$$\sqrt{1-\zeta^2}\,\omega_n t = n\pi, \quad n = 1, 2, 3, \cdots$$

from which we get

$$t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

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T_p occurs when $n = 1$, therefore

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \tag{3.9}$$

To determine M_p we substitute (3.9) back in (3.6)

$$\begin{aligned} y(t)\Big|_{t=T_p} &= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n\left(\pi/\omega_n\sqrt{1-\zeta^2}\right)} \sin\left(\omega_d\left(\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right)+\alpha\right) \\ &= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\pi/\sqrt{1-\zeta^2}} \sin(\pi+\alpha) \\ &= 1 + \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\pi/\sqrt{1-\zeta^2}} \sin(\alpha) \qquad (\text{noting that } \sin\alpha = \sqrt{1-\zeta^2}) \\ &= 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}} = M_p \end{aligned}$$

Therefore, since $y_{ss} = 1$, the percent overshoot is, from (3.8)

$$\% \text{ overshoot} = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

The percent overshoot is thus a function only of ζ and is plotted versus ζ in Figure 3.12. We can express (3.9) as

$$\omega_n T_p = \frac{\pi}{\sqrt{1-\zeta^2}}$$

Thus the product $\omega_n T_p$ is also a function of only ζ, and this is also plotted in Figure 3.12. Since T_p is an approximate indication of the rise time, Figure 3.12 also roughly indicates rise time. As ζ increases from 0 to 1, M_p, T_s, and the percent overshoot decreases, while T_p and T_r increase.

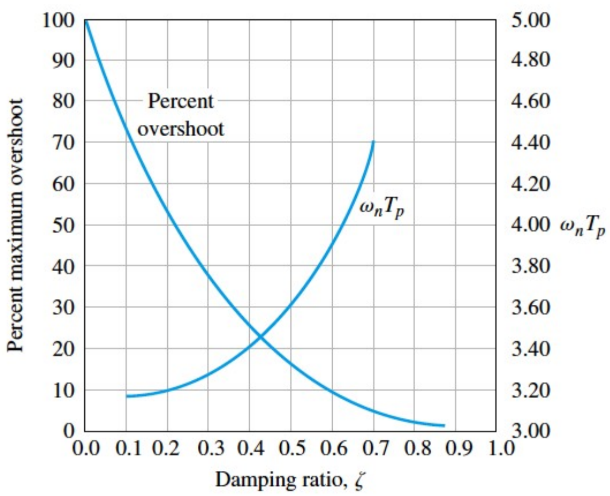

Figure 3.12: Percent overshoot and normalized peak time versus damping ratio ζ.

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Comment

The transient response of the system may be described in terms of two factors:

1. The **speed of response**, measured by the rise time, T_r, and the peak time, T_p.

2. The **tracking properties**, i.e., the closeness of the response to the desired response, is measured by the % overshoot and the settling time, T_s.

It is important to realize that the two factors are contradictory requirements; thus, a compromise must be obtained.

Example 3.3 Consider the following transfer function

$$G(s) = \frac{4}{s^2 + 2s + 4}$$

Determine, T_s, T_p and the % overshoot.

■ **Solution** We have

$$\omega_n^2 = 4 \implies \omega_n = 2 \text{ rad/s}, \quad 2\zeta\omega_n = 2 \implies \zeta = 0.5$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{3} \text{ rad/s}$$

The peak time is obtained as $T_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\sqrt{3}} = 1.82\text{s}$.

The settling time is found to be $T_s = \dfrac{4}{\zeta\omega_n} = 4\text{s}$.

The maximum % overshoot $= 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 16.3\%$

3.7 Specifications vs. pole locations

Figure 3.13 illustrates the relationships between the location of the characteristic equation roots and ζ, ω_n, and ω_d. For the complex conjugate roots shown,

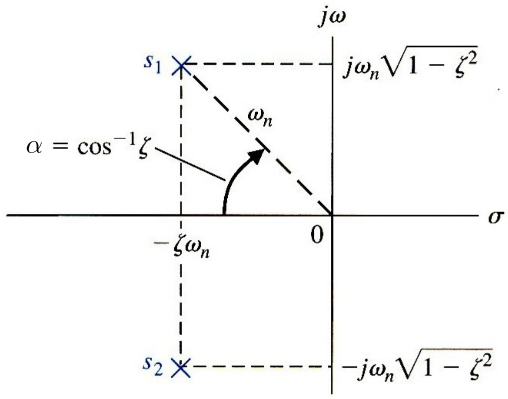

Figure 3.13: Relationship between poles of second-order system and ζ, ω_n, and ω_d.

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- ω_n is the radial distance from the roots to the origin of the s-plane.
- $\zeta\omega_n$ is the real part of the roots.
- ω_d is the imaginary part of the roots.
- ζ is the cosine of the angle between the radial line to the roots and the negative axis when the roots are in the left-halp s-plane, or $\zeta = \cos\alpha$.

The effect of increasing ω_n and ζ on pole locations in the s-plane is shown in Figure 3.14.

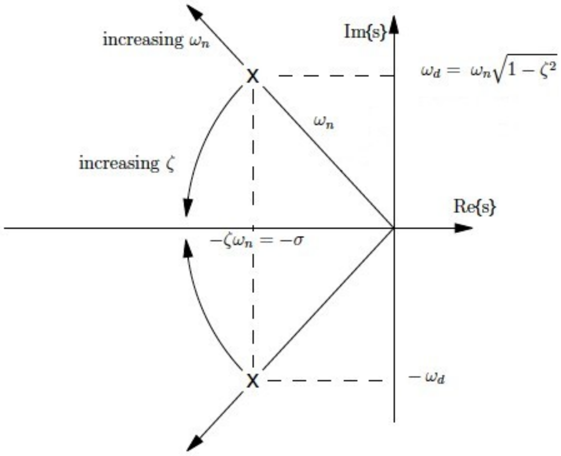

Figure 3.14: Pole locations tin the s-plane as ω_n and ζ increases respectively.

SETTLING TIME AND POLE LOCATIONS

The settling time T_s is related to the roots in Figure 3.14 by the relation

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma}$$

The settling time is then inversely related to the real parts of the poles. If in

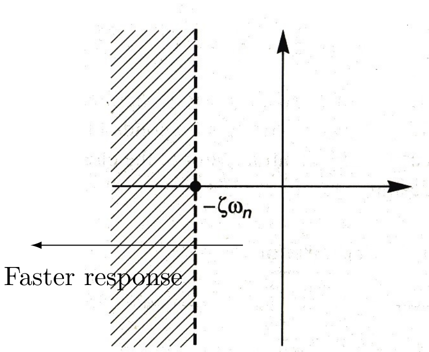

Figure 3.15: Pole locations to achieve desired settling time.

design the settling is specified to be less than or equal to some value $T_{s,desired}$,

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$\zeta\omega_n \geq 4/T_{s,desired}$, and the pole locations are then restricted to the region of the s-plane indicated in Figure 3.15. Hence the speed of response is increased by moving the poles to the left in the s-plane.

% Overshoot and pole locations

The angle α in Figure 3.13 satisfies the relationship

$$\alpha = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \cos^{-1}\zeta \tag{3.10}$$

The percent overshoot is given by

$$\%\ \text{overshoot} = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

Hence this equation can be expressed as

$$\%\ \text{overshoot} = e^{-\pi/\tan\alpha} \tag{3.11}$$

Decreasing the angle α reduces the percent overshoot. Hence, specifying the percent overshoot to be less than a particular value restricts the pole locations to the region of the s-planre, as shown in Figure 3.16

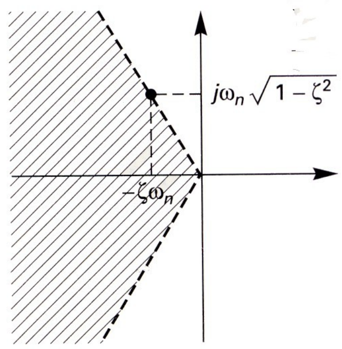

Figure 3.16: Pole locations and their relation to % overshoot.

Peak time and pole locations

From (3.9) the peak time T_p is related to the roots in Figure 3.14 by the relation

$$T_p = \frac{\pi}{\omega_d}$$

The peak time is inversely proportional to the imaginary part of the pole.

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Lines of constant T_p, % overshoot, and T_s are shown in Figure 3.17. Note that $T_{s_2} < T_{s_1}$; $T_{p_2} < T_{p_1}$; and $\%OS_1 < \%OS_2$.

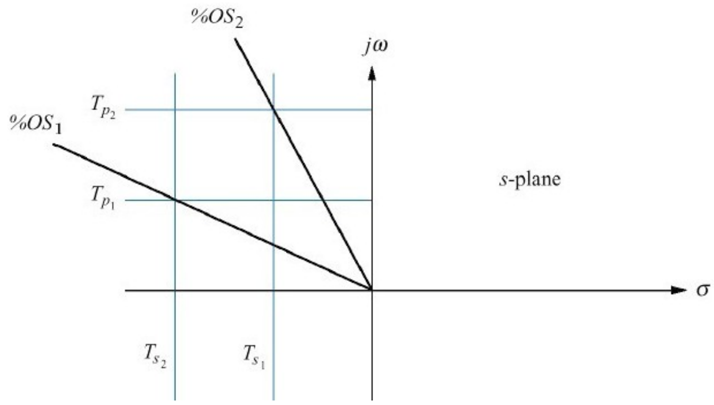

Figure 3.17: Lines of constant T_p, T_s, and % overshoot.

Example 3.4

Suppose that, in the design of a second-order system, the %overshoot in a step response is limited to 4.32%. A maximum settling time of 2s is also required. On an s-plane show the region to which the pole locations are limited to.

■ **Solution** To find ζ, from (3.11)

$$
\begin{aligned}
4.32 &= e^{-\pi/\tan\alpha} \times 100 \\
\implies 0.0432 &= e^{-\pi/\tan\alpha} \\
\implies 3.14 &= \frac{\pi}{\tan\alpha} \qquad \text{after taking natural log of both sides} \\
\therefore \tan\alpha &= 1
\end{aligned}
$$

Thus, $\alpha = 45^o$, hence $\zeta = \cos 45^o = 0.707$. A settling time of 2s implies $\zeta\omega_n \geq 2$. Hence the pole locations are limited to the regions of the s-plane shown in Figure 3.18. The pole locations that exactly satisfy the limits of the specifications are $s = -2 \pm j2$. ■

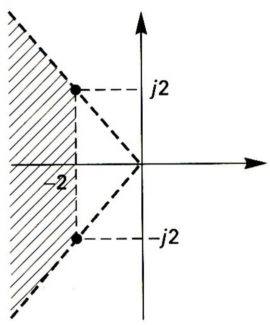

Figure 3.18: Pole plot for Example 3.4.

Example 3.5

Consider the pole plot shown in Figure 3.19. Find ζ, ω_n, T_p, and T_s.

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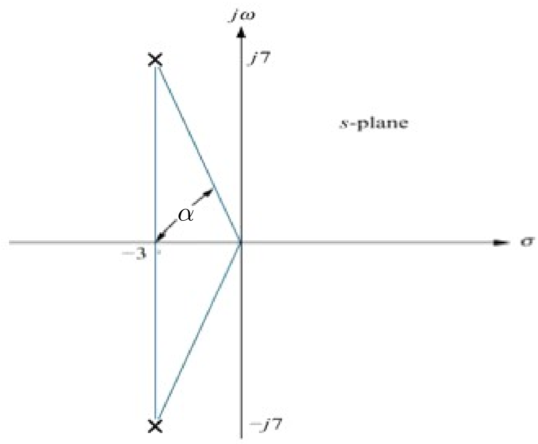

Figure 3.19: Pole plot for Example 3.5.

■ **Solution** The damping ratio is given by

$$\zeta = \cos\alpha = \cos\left[\tan^{-1}(7/3)\right] = 0.394$$

The natural frequency ω_n is given by $\omega_n = \sqrt{7^2 + 3^2} = 7.616$ rad/s

The peak time T_p is $T_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{7} = 0.449$s

The settling time T_s is $T_s = \dfrac{4}{\zeta\omega_n} = \dfrac{4}{3} = 1.33$s ■

3.7.1 Step response vs. pole location

In Figure 3.20 the step responses are shown as the poles are moved in vertical direction, keeping the real part the same. We see that the frequency changes, but the envelope remains the same. Since all curves fit under the same exponential decay curve, the settling time is virtually the same for all waveforms.

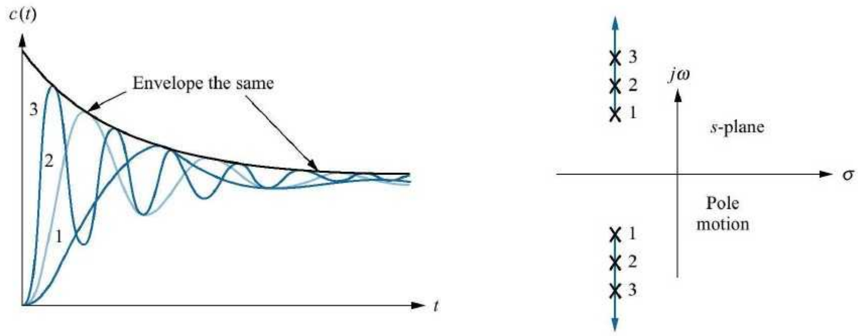

Figure 3.20: Step response as poles move with constant real part.

In Figure 3.21 the step responses are shown as the poles are moved in horizontal

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direction, keeping the imaginary part the same. As the poles move to the left, the response damps out rapidly, while the frequency remains the same. Notice that the peak time is the same for all waveforms.

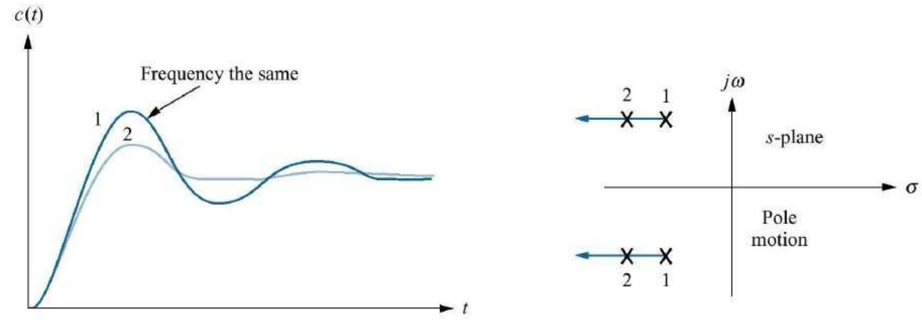

Figure 3.21: Step response as poles move with constant imaginary part.

In Figure 3.22 the poles are moved along a constant radial line. We see that the % overshoot remains the same. The farther the poles are from origin, the more rapid the response.

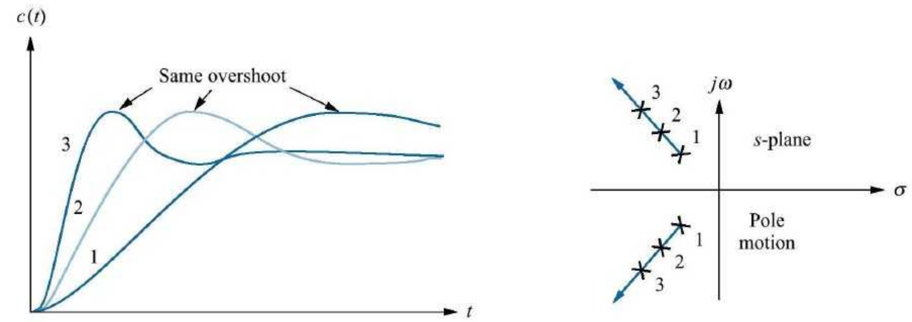

Figure 3.22: Step response as poles move with constant damping ratio.

3.8 Steady-state accuracy

It has been observed, in the previous sections, that much information about a system can be obtained from the analysis of its response to test inputs. In control system design, one of the major objectives is to be able to track reference inputs precisely, and to maintain this precision in the face of disturbances. In many cases an error between the desired and resulted final value does occur. In this section, we consider how such precision can be achieved.

3.8.1 Error signal

Consider the feedback system in Figure 3.23. Assume the feedback loop is stable. We have seen in Section 1.3 how a feedback control system can reduce

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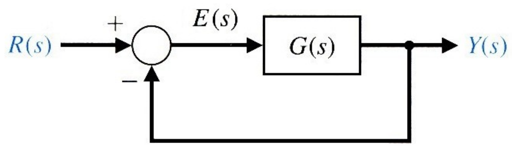

Figure 3.23: Closed loop control system.

sensitivity of the system. Furthermore, the effect of disturbances can also be reduced significantly. However, as a further requirement, we must examine and compare the final steady-state error of the closed loop system.

The output $Y(s)$ is required to track the reference input $R(s)$. The input into plant $G(s)$ is the tracking error

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= \frac{R(s)}{1+G(s)} \end{aligned}$$

Let $e(t)$ denote its inverse Laplace transform. The limit

$$e_{ss} = \lim_{t\to\infty} e(t)$$

is referred to as the **steady-state error**. The value of e_{ss} characterises the final value of error as a difference between the value of the input $r(t)$ and the final value of the output $y(t)$. In other words, e_{ss} is the value of error after transients have died out.

To calculate the steady-state error, we utilize the final value theorem, as long as $E(s)$ does not have any poles in the right half of the s-plane, except maybe, at $s = 0$, then

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Step Input The steady-state error for a step input is

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \frac{1}{1+G(0)}$$

Note that in this case, the steady-state error is determined by the dc gain of $G(s)$. The larger is dc gain, the smaller is the steady-state error. Furthermore, if $G(s)$ has one or more poles at $s = 0$, then $\lim_{s\to 0} G(s) = \infty$. In this case, $e_{ss} = 0$. A pole at $s = 0$ implies $G(s)$ includes an integrator. **The number of poles at the origin of the loop gain (i.e. the number of integrators) defines the system's type number, *N*.**

Type zero systems ($N = 0$) do not include an integrator and therefore have a finite dc gain $G(0)$. The constant $G(0)$ is denoted by K_p, the **position error constant**, and is given by

$$K_p = \lim_{s\to 0} G(s)$$

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The steady-state error for a step input is thus given by

$$e_{ss} = \frac{1}{1+K_p}$$

Note that a large position error constant corresponds to a small steady-state error. Furthermore, if $N \geq 1$ the steady-state error $e_{ss} = 0$.

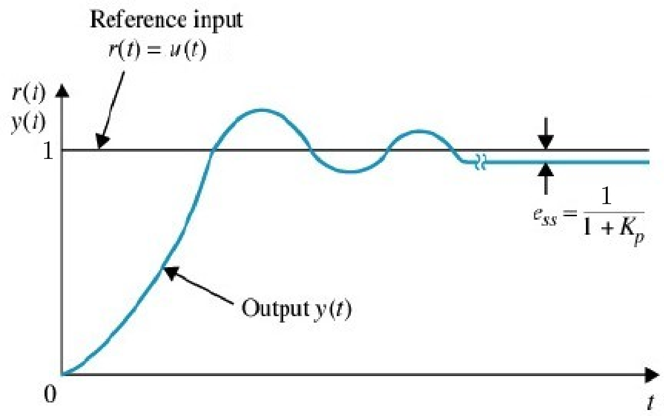

Figure 3.24: Steady-state error for step input.

Ramp Input Now consider the steady -state error in response to a ramp input $r(t) = t$. The Laplace transform of the ramp input is $R(s) = 1/s^2$ and

$$e_{ss} = \lim_{s\to 0} \frac{s(1/s^2)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \lim_{s\to 0} \frac{1}{sG(s)}$$

Again, the steady-state error depends upon the number of integrators, N. For a type zero system, $N = 0$, the steady-state error is infinite. For a type one system, $N = 1$, the error is

$$e_{ss} = \frac{1}{K_v}$$

where K_v the **velocity error constant** and is computed as

$$K_v = \lim_{s\to 0} sG(s)$$

For $N \geq 2$, K_v is infinite and the steady-state error for a ramp input is zero.

Parabolic Input for a parabolic input $r(t) = t^2/2$, we take $R(s) = 1/s^3$, the steady-state error is

$$e_{ss} = \lim_{s\to 0} \frac{s(1/s^3)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2G(s)}$$

The steady-sate error is infinite for type zero and type one systems. For type two, $N = 2$, and we obtain

$$e_{ss} = \frac{1}{K_a}$$

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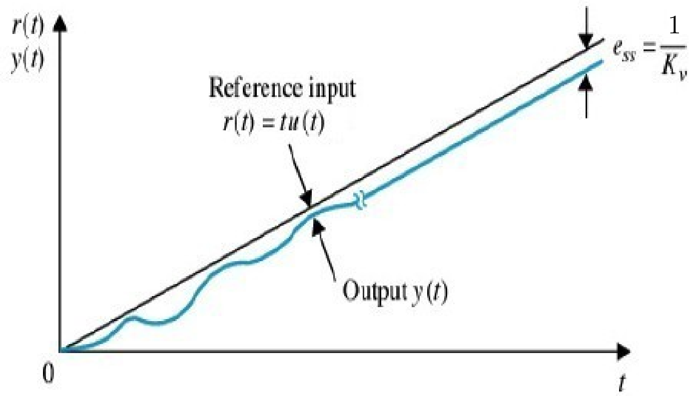

Figure 3.25: Steady-state error for ramp input.

where K_a is called the **acceleration error constant**. The acceleration error constant is

$$K_a = \lim_{s\to 0} s^2 G(s)$$

The error constants and the steady-state error for the three inputs are summarized in Table 3.1.

Table 3.1: Summary of steady-state errors.

Type	$R(s) = \frac{1}{s}$	$R(s) = \frac{1}{s^2}$	$R(s) = \frac{1}{s^3}$	*Error constants*
0	$e_{ss} = \frac{1}{1+K_p}$	∞	∞	$K_p = \lim_{s\to 0} G(s)$
1	$e_{ss} = 0$	$\frac{1}{K_v}$	∞	$K_v = \lim_{s\to 0} sG(s)$
2	$e_{ss} = 0$	0	$\frac{1}{K_a}$	$K_a = \lim_{s\to 0} s^2G(s)$

Example 3.6

Consider the control system in Figure 3.23 with

$$G(s) = \frac{200(s+1)^2}{(s+2)(s+3)(s+4)}$$

Find the steady-state error when (a) $r(t)$ is a unit step, (b) $r(t)$ is a unit ramp.

■ **Solution** $G(s)$ is a type zero system, therefore for a step input, the position error constant, $K_p = G(0) = 200/24 = 8.333$. Then

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{9.333} = 0.1071$$

For a ramp input $K_v = \lim_{s \to 0} sG(s) = 0$. Then,

$$e_{ss} = \frac{1}{K_v} = \infty \qquad \blacksquare$$