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Chapter 4

Stability Analysis

Stability is the most crucial issue in designing any control system. One of the most common control problems is the design of a closed loop system such that its output follows its input as closely as possible. If the system is unstable such behavior is not guaranteed. Unstable systems exhibit an unbounded output, i.e., a response blowing up to infinity as time increases. This usually cause the system to suffer serious damage such as burn out, break down or it may even explode. Therefore, for such reasons our primary goal is to guarantee stability. As soon as stability is achieved one seeks to satisfy other design requirements, such as speed of response, settling time, steady state error, etc.

4.1 Introduction

To help make the later mathematical treatment of stability more intuitive let us begin with a general discussion of stability concepts and equilibrium points. Consider the ball which is free to roll on the surface shown in Figure 4.1. The ball could be made to rest at points A, E, F, and G and anywhere between points B and D, such as at C. Each of these points is an equilibrium point of the system.

A small perturbation away from points A or F will cause the ball to diverge from these points. This behavior justifies labeling points A and F as *unstable*

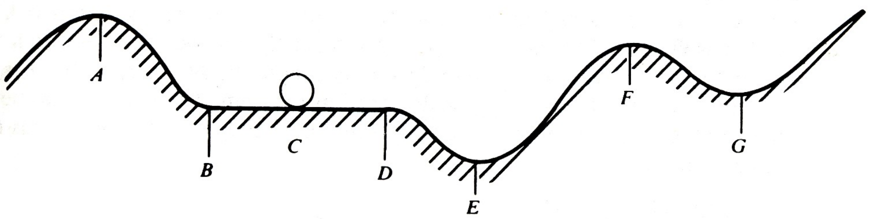

Figure 4.1: Equilibrium points

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equilibrium points. After small perturbations away from E and G, the ball will eventually return to rest at these points. Thus E and G are labeled as stable equilibrium points. If the ball is displaced slightly from point C, it will normally stay at the new position. Points like C are sometimes said to be *neutrally* stable.

So far we assumed small perturbations, if the ball was displaced sufficiently far from point G, it would not return to that point. We say the system is stable *locally.* Stability therefore depends on the size of the original perturbation and on the nature of any disturbances.

Stability deals with the following questions. If at time t_0 the system is perturbed from its equilibrium point, does the system return to that point, or remain close to it, or diverge from it?

As we shall see in this chapter, stability of a feedback system is directly related to the location of the roots of the characteristic equation of the system transfer function.

4.2 Bounded-input bounded-output stability

A continuous-time system is stable if and only if every bounded input produces a bounded output. Consider a bounded input $x(t)$ such that $|x(t)| < B$ for all t. Suppose that this input is applied to an LTI system with impulse response $h(t)$. Then

$$\begin{aligned} |y(t)| &= \left| \int_0^\infty h(\tau)x(t-\tau)d\tau \right| \\ &\leq \int_0^\infty |h(\tau)|\,|x(t-\tau)|\,d\tau \\ &\leq B\int_0^\infty |h(\tau)|\,d\tau \end{aligned}$$

Therefore, because B is finite, $y(t)$ is bounded, hence, the system is stable if

$$\int_0^\infty |h(\tau)|\,d\tau < \infty \tag{4.1}$$

Example 4.1

For an LTI system with impulse response $h(t) = e^{-3t}u(t)$, determine the stability of this causal LTI system.

■ **Solution** Using (4.1), hence

$$\int_0^\infty e^{-3t}dt = -\frac{1}{3}e^{-3t}\Big|_0^\infty = \frac{1}{3} < \infty$$

and this system is stable. ■

4.2.1 Relationship between characteristic equation roots and stability

To show the relation between the roots of the characteristic equation and stability, let $G(s)$ be a transfer function representation of an LTI system. Then,

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$G(s)$ can be written as

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

where the denominator polynomial is called the **characteristic polynomial** of $G(s)$. The equation

$$a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \tag{4.2}$$

is called the **characteristic equation** of $G(s)$. The roots of the characteristic equation are called the **poles** of $G(s)$. The **order** of the system $G(s)$ is defined to be the degree of the characteristic polynomial.

Assume that the roots p_i of the characteristic equation are real or complex, but are distinct. The solution to the differential equation whose characteristic equation is given by (4.2) may be written using partial-fraction expansion as

$$y(t) = \sum_{i=1}^{n} K_i e^{p_i t} \tag{4.3}$$

where p_i are the roots of (4.2). The system is stable if and only if every term in (4.3) goes to zero as $t \to \infty$:

$$e^{p_i t} \to 0 \qquad \text{for all } p_i$$

This will happen if all the poles of the system are strictly in the LHP, where

$$\text{Re}\{p_i\} < 0$$

If any poles are repeated, the response must be changed from that of (4.3) by including a polunomial in t in place of K_i, but the conclusion is the same.

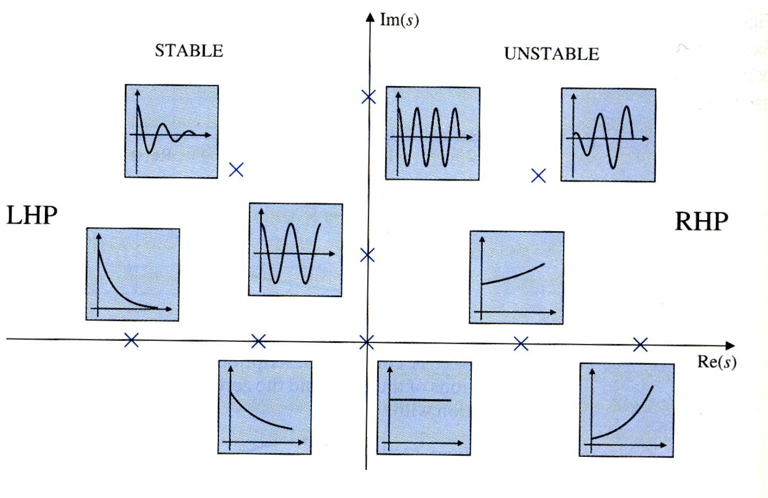

Figure 4.2: Time functions associated with pole locations in the s-plane.

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Therefore, the stability of a system can be determined by computing the location of the roots of the characteristic equation and determining whether they are all in the LHP. If the system has any poles in the RHP, it is **unstable**. Hence the $j\omega$ axis is the stability boundary between BIBO stable and unstable responses. A system is said to be **marginally stable** if all its poles lie in the LHP, and at least one pole on the $j\omega$ axis. If the system has repeated poles on the $j\omega$ axis, then it is **unstable**. For example, repeated poles at $s = 0$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$$

results in an unbounded response. The following table illustrates the stability conditions of an LTI system with reference to the locations of the roots of the characteristic equation. Figure 4.2 summarizes all the results.

Table 4.1: Stability conditions of a LTI system.

Stability conditions	*Locations of the roots*
Stable	All the roots are in the LHP
Marginally stable	At least one root and no multiple roots on the $j\omega$-axis; and no roots in the RHP.
Unstable	At least one root in the RHP or at least one multiple roots on the $j\omega$-axis.

The following examples illustrates the stability conditions of systems with reference to the poles of the transfer function $G(s)$.

Table 4.2: Stability examples.

Transfer function $G(s)$	*Stability condition*
$G(s) = \dfrac{20}{(s+1)(s+2)(s+3)}$	Stable
$G(s) = \dfrac{20(s+1)}{(s-1)(s^2+2s+2)}$	Unstable due to the pole at $s = 1$
$G(s) = \dfrac{20(s-1)}{(s+2)(s^2+4)}$	Marginally stable due to $s = \pm j2$
$G(s) = \dfrac{10}{(s^2+4)^2(s+10)}$	Unstable due to the multiple poles at $s = \pm j2$

As shown above, one way of determining stability is to calculate the roots of the characteristic equation. The disadvantage of this is that system parameters must be assigned numerical values, which makes it difficult to find the

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range of values of a parameter that ensures stability. For example in the spring-mass-damper system the characteristic equation is $Ms^2 + Bs + K$, one can not determine the ranges of M, B and K to ensure stability.

An alternative to locating the roots of the characteristic equation is given by Routh-Hurwitz stability criterion, which is presented next.

4.3 Routh-Hurwitz Stability Criterion

4.3.1 General properties of polynomials

Consider the second order polynomial, assuming all coefficients are real

$$P(s) = s^2 + a_1 s + a_0 = (s - p_1)(s - p_2) \tag{4.4}$$
$$= s^2 - (p_1 + p_2)s + p_1 p_2 \tag{4.5}$$

By comparing (4.4) with (4.5) implies

$$a_1 = -(p_1 + p_2) \quad \text{and} \quad a_0 = p_1 p_2$$

If p_1 and p_2 are stable, we have $a_1 > 0$ and $a_0 > 0$. Consider next the third order polynomial

$$\begin{aligned} P(s) &= s^3 + a_2 s^2 + a_1 s + a_0 = (s - p_1)(s - p_2)(s - p_3) \\ &= s^3 - (p_1 + p_2 + p_3)s^2 + (p_1 p_2 + p_2 p_3 + p_1 p_3)s - p_1 p_2 p_3 \end{aligned}$$

and by comparing the two equations we have

$$\begin{aligned} a_2 &= -(p_1 + p_2 + p_3) \\ a_1 &= (p_1 p_2 + p_2 p_3 + p_1 p_3) \\ a_0 &= -p_1 p_2 p_3 \end{aligned}$$

Again if all roots are stable, all the coefficients will have the same sign. **However, this condition is not sufficient, for it is quite possible that an equation with all its coefficients nonzero and of the same sign still will not have all the roots in the left half of the s-plane**. Consider for example the polynomial $s^3 + s^2 + 2s + 8$, clearly all the coefficients have the same sign however not all roots are in the LHP.

Consider a general n^{th} order polynomial

$$\begin{aligned} P(s) &= s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \\ &= (s - p_1)(s - p_2) \cdots (s - p_n) \end{aligned}$$

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By analogy with the previous examples, we have

$$
\begin{aligned}
a_{n-1} &= -\sum_{i=1}^{n} p_i \\
a_{n-2} &= \sum \text{ product of roots taken 2 at a time} \\
a_{n-3} &= -\sum \text{ product of roots taken 3 at a time} \\
&\vdots \\
a_0 &= (-1)^n \prod_{i=1}^{n} p_i
\end{aligned}
$$

We conclude

- If all roots are stable, all the polynomial coefficients will be positive.
- If any coefficient is negative, at least one root is unstable.
- If any coefficient is zero, not all roots are stable.

4.3.2 Routh-Hurwitz stability criterion

The Routh-Hurwitz criterion represents an analytical procedure of determining if all roots of a polynomial with constant real coefficients lie in the left half of the s-plane, without actually solving for the roots.

Consider the n^{th} order polynomial

$$P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

in which we can always assume $a_0 \neq 0$. If $a{+}_0 = 0$, we can write:

$$P(s) = s\underbrace{\left(a_n s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1\right)}_{\hat{P}(s)}$$

and work with $\hat{P}(s)$ instead. Note that in the case $a_0 = 0$, $P(s)$ will have at least one root at the origin and we conclude that the LTI system is marginally stable or unstable.

The Routh array

The first step in the Routh-Hurwitz criterion is to arrange the coefficients of the characteristic polynomial into an array as follows

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots
\end{array}
$$

Further rows are then completed as

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$$\begin{array}{c|cccccc} s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\ s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\ s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s^2 & k_1 & k_2 & & & \\ s^1 & l_1 & & & & \\ s^0 & m_1 & & & & \end{array}$$

where

$$b_1 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} \qquad b_2 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_1 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} \qquad c_2 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}$$

$$\vdots$$

$$m_1 = -\frac{1}{l_1} \begin{vmatrix} k_1 & k_2 \\ l_1 & 0 \end{vmatrix}$$

Once the Routh's array has been completed, we investigate the signs of the coefficients in the first column of the array. The roots of the equation are all in the left half of the s-plane if all the elements of the first column of the Routh's array are of the same sign. *The number of unstable roots is equal to the number of sign changes in the first column of the array.*

Example 4.2

Consider the third order polynomial:

$$P(s) = s^3 - s^2 + s + 6$$

This equation has one negative coefficient. Thus, we know without applying Routh's test that not all the roots of the equation are in the LHP. ■

Example 4.3

Consider the third order polynomial:

$$P(s) = s^3 + s^2 + 2s + 8$$

The Routh array is:

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 8 \\ s & -6 & \\ 1 & 8 & \end{array}$$

The two sign changes (from +1 to -6 and from -6 to +8) indicates two unstable roots. ■

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SPECIAL CASES WHEN THE ROUTH'S ARRAY TERMINATES PREMATURELY

Not all arrays can be completed as the one shown in Example 4.3. Depending on the coefficients of the equation, the following difficulties may occur that prevent the Routh's array from completing properly:

1. *The first element of a row is zero, with at least one nonzero element in the same row, the procedure is modified by replacing that first element with a small number ϵ such that $|\epsilon| \ll 1$ and proceeding as before.*

2. *Every entry in a row is zero, the last modification will not give useful information and another modification is needed.*

Example 4.4 To demonstrate the first case consider the polynomial:

$$P(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

The Routh array is then

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & 0 & 6 & 0 \end{array}$$

Since the first element of the s^3 row is zero, the elements in the s^2 row would all be infinite. To overcome this difficulty, we replace the zero in the s^3 row by a small positive number ϵ and then proceed with the array

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & 0 \to \epsilon & 6 & 0 \end{array}$$

The first element of the s^2 row is calculated as follows

$$-\frac{1}{\epsilon}\begin{vmatrix} 2 & 4 \\ \epsilon & 6 \end{vmatrix} = -\frac{1}{\epsilon}(12 - 4\epsilon) = 4 - \frac{12}{\epsilon} \backsimeq -\frac{12}{\epsilon}$$

with a similar procedure we calculate the remaining rows

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & 0 \to \epsilon & 6 & 0 \\ s^2 & -12/\epsilon & 10 & \\ s & 6 & & \\ 1 & 10 & & \end{array}$$

There are two sign changes (irrespective of the sign of ϵ) indicating two unstable roots. ■

Special Case: Zero rows. If all the coefficients in a row are zero, a pair of roots of equal magnitude and opposite sign is indicated. These could be two real roots with equal magnitudes and opposite signs or two conjugate imaginary

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roots. The zero row is replaced by taking the coefficients of $dP_a(s)/ds$, where $P_a(s)$, called the **auxiliary polynomial**, is obtained from the values in the row above the zero row. Why? A zero row implies that a polynomial $P_a(s)$ has only even or odd powers. It turns out in this case, $P_a(s)$ and $P_a(s)+dP_a(s)/ds$ have exactly the same number of RHP poles (proof beyond the scope of theis course). As the goal is just to find the number of RHP poles, we can use $dP_a(s)/ds$ as a surrogate to continue the procedure. The pair of roots can be found by solving $dP_a(s)/ds = 0$. The roots of $P_a(s)$ are also the roots of the the ploynomial $P(s)$.

Example 4.5

As an example of case 2 consider the the polynomial:

$$P(s) = s^3 + s^2 + 2s + 2$$

The Routh array is then

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s & 0 & 0 \end{array} \quad \begin{array}{l} \\ \longleftarrow \text{auxiliary polynomial } P_a(s) = s^2 + 2 \\ \\ \end{array}$$

The auxiliary polynomial $P_a(s)$ indicates that $P(s) = 0$ must have two pairs of roots of equal magnitude and opposite sign, which are also roots of the auxiliary polynomial equation $P_a(s) = 0$. Taking the derivative of $P_a(s)$ with respect to s we obtain

$$\frac{dP_a(s)}{ds} = 2s$$

so the s row is as shown below and the Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s & 2 & 0 \\ 1 & 2 & \end{array} \quad \begin{array}{l} \\ \\ \longleftarrow \text{Coefficients of } dP_a(s)/ds \\ \\ \end{array}$$

The absence of a sign change indicates no unstable roots, so all roots are on the imaginary axis. We conclude the system is marginally stable. ■

Example 4.6

The following example combines case 1 and case 2 problems: polynomial:

$$P(s) = s^4 + 4$$

The Routh array is then

$$\begin{array}{c|ccl} s^4 & 1 & 0 & 4 \longleftarrow P_a(s) = s^4 + 4 \\ s^3 & 0 \to 4 & 0 & 0 \longleftarrow \text{Coefficients of } \dfrac{dP_a(s)}{ds} = 4s^3 \\ s^2 & 0 \to \epsilon & 4 & \\ s & -16/\epsilon & & \\ 1 & 4 & & \end{array}$$

The two sign changes indicate two unstable roots. ■

In Summary, the three cases that occur in the application of the Routh-Hurwitx criterion are as follows:

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Case 1. No elements in the first column are zero. There are no problems in completing the array.

Case 2. There is at least one nonzero element in a row, with the first element equal to zero. This always indicates an unstable system. The first element (which is zero) is replaced with the value ϵ, $\epsilon \ll 1$, and the calculation of the array continues.

Case 3. All elements in a row are zero. This always indicates a system that is not stable, but it may be marginally stable. This case can be analyzed through the use of the auxiliary equation, as described earlier. If the system is marginally stable the roots on the $j\omega$ axis are also the roots of the auxiliary equation.

4.4 Applications in feedback design

Consider the following feedback system involving a plant $G(s)$ and a compensator (or controller) $K(s)$

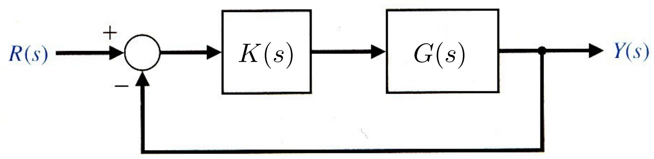

Figure 4.3: Feedback control system.

The closed-loop transfer function $H(s)$ is

$$H(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

The closed-loop poles are given by the roots of the **charactersistic equation**

$$1 + G(s)K(s) = 0$$

Suppose we write

$$G(s) = \frac{N_g(s)}{D_g(s)} \qquad \text{and} \qquad K(s) = \frac{N_k(s)}{D_k(s)}$$

where $N_g(s)$, $D_g(s)$, $N_k(s)$, and $D_k(s)$ are all polynomials. Then, the closed loop transfer function is given

$$H(s) = \frac{\dfrac{N_g(s)N_k(s)}{D_g(s)D_k(s)}}{1 + \dfrac{N_g(s)N_k(s)}{D_g(s)D_k(s)}}$$

$$= \frac{N_g(s)N_k(s)}{N_g(s)N_k(s) + D_g(s)D_k(s)}$$

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The poles of the closed-loop system are also given by the roots of the characteristic polynomial

$$N_g(s)N_k(s) + D_g(s)D_k(s)$$

The poles of the open-loop system $G(s)$ are the roots of its characteristic polynomial

$$D_g(s) = 0$$

which are generally different from the closed-loop poles.

A fundamental design objective in control is the stabilization of unstable systems. The Routh-Hurwitz stability criterion can be used as an aid in feedback design.

Example 4.7

Let $G(s)$ the be given by

$$G(s) = \frac{1}{s^3 + 5s^2 + 2s - 8}$$

Let $K(s) = K$ be a constant controller. Find the range of values of K for which the closed-loop is stable.

■ **Solution** Since the coefficients of the characteristic polynomial do not have the same sign, we conclude $G(s)$ is unstable. The closed-loop poles are the roots of the characteristic equation

$$1 + KG(s) = 1 + \frac{K}{s^3 + 5s^2 + 2s - 8} = 0$$

which implies

$$s^3 + 5s^2 + 2s + (K - 8) = 0$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 5 & K-8 \\ s^1 & 0.2(18-K) & \\ 1 & K-8 & \end{array}$$

For stability we need

$$8 < K < 18 \qquad ■$$

Example 4.8

Let $G(s)$ the be given by

$$G(s) = \frac{1}{s^3 - s^2 - 10s - 8}$$

Find the range of values of K for which the closed-loop is stable.

■ **Solution** The characteristic equation is given by

$$1 + KG(s) = 1 + \frac{K}{s^3 - s^2 - 10s - 8} = 0$$

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which implies

$$s^3 - s^2 - 10s + (K - 8) = 0$$

It follows that no choice of K can ensure all coefficients have the same sign. We conclude that $G(s)$ cannot be stabilized by a constant controller and **dynamic compensation is need**. ■