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Chapter 5

Root-Locus Analysis and Design

In this chapter we introduce one of the major analysis and design methods discussed in this course. The method is the root-locus procedure; it indicates to us the characteristics of a control system's transient response. We have seen that the response of an LTI system is largely determined by the location of its poles.

5.1 Root-locus principles

We introduce the root locus through an example. Consider the feedback control system shown in Figure 5.1

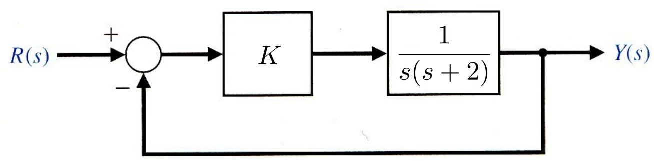

Figure 5.1: Feedback control system.

The closed-loop transfer function is given by

$$H(s) = \frac{\dfrac{K}{s(s+2)}}{1 + \dfrac{K}{s(s+2)}} = \frac{K}{s^2 + 2s + K}$$

Hence the characteristic equation, which is the denominator of the closed-loop transfer function set to zero, is

$$s^2 + 2s + k = 0$$

We see, since the polynomial is second order, that the system is stable for all positive values of K. It is not evident for this example exactly how the value of

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Table 5.1: Characteristic equation roots for different values of K.

K	*Characteristic equation*	*Roots*	
0	$s^2 + 2s = 0$	$s = 0, -2$	Note these are the open loop poles
1	$s^2 + 2s + 1 = 0$	$s = -1 \pm j0$	
2	$s^2 + 2s + 2 = 0$	$s = -1 \pm j$	

K affects the transient response. Table 5.2 show the roots of the characteristic equation for different values of K. To investigate some of the effects of choosing different values of K, we plot the roots of the system characteristic equation in the s-plane. These roots are plotted in Figure 5.2 for $0 \leq K \leq \infty$.

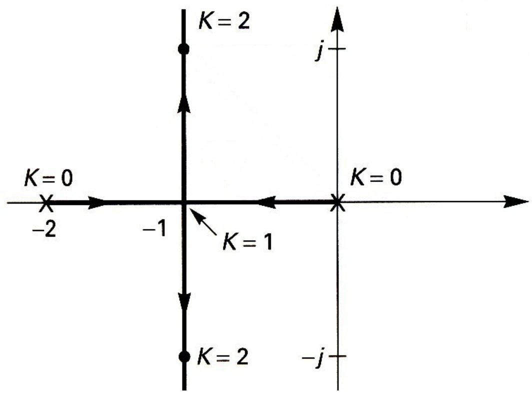

Figure 5.2: Plot of characteristic equation roots.

We can see from the plot that for $0 < K < 1$, the roots are real with different time constants. For $K = 1$, the roots are real and equal, and the system is critically damped. For $K > 1$, the roots are complex with a time constant of 1s, with the value of ζ decreasing as K increases. Hence, as K increases with the roots being complex, the overshoot in the transient response increases.

The plot in Figure 5.2 is called the **root locus** of the system in Figure 5.1. **The root-locus of a system is a plot of the roots of the system characteristic equation (closed-loop poles) as K varies from 0 to ∞.**

For an n^{th} order system, the root-locus is a family of n curves traced out by the n closed-loop poles as K is varied from zero to infinity. Plotting the root locus for negative values of K will be considered later.

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5.1.1 Root-locus criterion

We generally consider the system of Figure 5.3 in discussing the root locus, with $0 \leq K < \infty$.

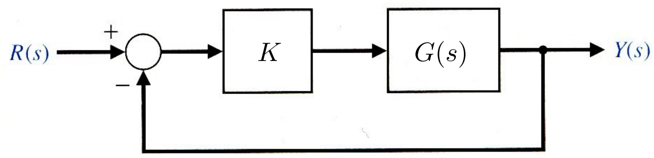

Figure 5.3: Feedback control system.

The characteristic equation for this system is given by

$$1 + KG(s) = 0 \tag{5.1}$$

A point s_1 lies on the root-locus if and only if s_1 satisifies (5.1) for a real value of K, with $0 \leq K < \infty$. Equation (5.1) can be written as

$$K = -\frac{1}{G(s)} \tag{5.2}$$

$$G(s) = -\frac{1}{K} \tag{5.3}$$

$$KG(s) = -1 \tag{5.4}$$

Equations (5.1) to (5.4) are all equivalent.

The magnitude citerion

Since K is real and positive, (5.2) is equivalent to

$$K = \frac{1}{|G(s)|} \tag{5.5}$$

We call (5.5) the **magnitude criterion** of the root locus and can be used to find K corresponding to a point on the root-locus.

Example 5.1

Assume the point $s_1 = -2$ lies on the root-locus find K if

$$G(s) = \frac{s+4}{(s+1)(s+3)}$$

■ **Solution** Evaluate $|G(s)|_{s=-2}$, we have

$$|G(-2)| = \frac{|-2+4|}{|-2+1||-2+3|} = 2$$

Hence,

$$K = \frac{1}{|G(-2)|} = 0.5$$

The value $K = 0.5$ can be interpreted as the gain needed, in the feedback control system of Figure 5.3, that places the locus at the point $s = -2$. ■

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The angle criterion

In general $G(s)$ is complex and can be expressed in polar form as magnitude and phase as $|G(s)|\underline{/G(s)}$. If $G(s)$ is to satisfy (5.4) we note the following

$$\underline{/KG(s)} = 180^\circ$$

but K is real and positive which means $\underline{/K} = 0$, hence,

$$\underline{/G(s)} = 180^\circ \tag{5.6}$$

and in general

$$\underline{/G(s)} = \pm r(180^\circ) \qquad r = 1, 3, 5, \cdots$$

Equation (5.6) called the **angle criterion** may be interpreted as follows: *For a point s_1 to be on the root-locus, the sum of all angles for vectors between open-loop poles and zeros to point s_1 must be equal* 180°. The angle criterion is illustrated in Figure 5.4 for the function

$$G(s) = \frac{s - z_1}{(s - p_1)(s - p_2)}$$

In Figure 5.4 the poles of $G(s)$ are marked $\times$ and the zero is marked $\bigcirc$. Suppose that the point s_1 is to be tested to determine if it is is on the root-locus. For this point to be on the locus, we must have $\underline{/G(s_1)} = \pm 180^\circ$ or equivalently

$$\underline{/(s_1 - z_1)} - \underline{/(s_1 - p_1)} - \underline{/(s_1 - p_2)} = \pm r(180^\circ) \qquad r = 1, 3, 5, \cdots$$

The angle from the zero term $s - z_1$ can be computed by drawing a line from the location of the zero at z_1 to the test point s_1. In this case the line has a phase angle marked θ_1 on Figure 5.4. In a similar fashion, the vector from the pole $s = p_1$ to the test point s_1 is shown with angle θ_2, and the angle of the vector from the pole $s = p_2$ to s_1 is shown with angle θ_3. Thus the angle condition (5.6) becomes

$$\theta_1 - \theta_2 - \theta_3 = \pm r(180^\circ)$$

for the point s_1 to be on the root-locus.

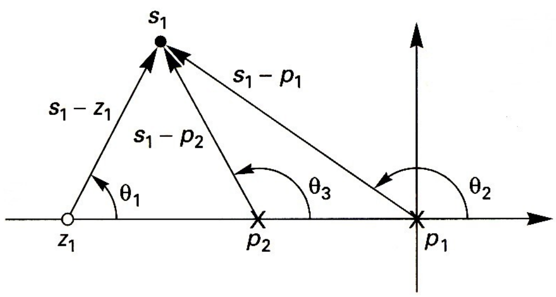

Figure 5.4: Illustration of the angle criterion.

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In general the condition for a point in the s-plane to be on the root-locus is that

$$\sum_i (\text{angles form } z_i) - \sum_i (\text{angles from } p_i) = \pm r(180^\circ) \qquad r = 1, 3, 5, \cdots$$

Example 5.2

Check whether the point $s_0 = -1 + 2j$ lies on the root-locus for some value of K if

$$G(s) = \frac{s+1}{s[(s+2)^2+4](s+5)}$$

■ **Solution** For s_0 to be on the locus, we must have $\angle G(s_0) = \pm 180^\circ$. Therefore,

$$\begin{aligned}
\angle G(s_0) &= \angle(s_0+1) - \angle s_0 - \angle[(s_0+2)^2+4] - \angle(s_0+5) \\
&= \angle(s_0+1) - \angle s_0 - \angle(s_0+2-2j) - \angle(s_0+2+2j) - \angle(s_0+5) \\
&= \angle(2j) - \angle(-1+2j) - \angle(1) - \angle(1+4j) - \angle(4+2j) \\
&= 90^\circ - 116.6^\circ - 0^\circ - 76^\circ - 26.6^\circ \\
&= -129.2^\circ
\end{aligned}$$

Alternatively, we could have marked the poles and zeros of $G(s)$ on the s-plane as shown in Figure 5.5. The angles from the poles and zeros could be computed by drawing a line from each pole and zero to the test point s_0 as shown in Figure 5.5. The point s_0 is on the root locus if

$$\sum_i (\text{angles form } z_i) - \sum_i (\text{angles from } p_i) = 180^\circ$$

and by inspecting Figure 5.5 yields

$$\begin{aligned}
\angle G(s_0) &= \psi_1 - \phi_1 - \phi_2 - \phi_3 - \phi_4 \\
&= 90^\circ - 116.6^\circ - 0^\circ - 76^\circ - 26.6^\circ \\
&= -129.2^\circ
\end{aligned}$$

Since the phase of $G(s_0)$ is not 180°, we conclude that s_0 is not on the root-locus, so we must select another point and try again. ■

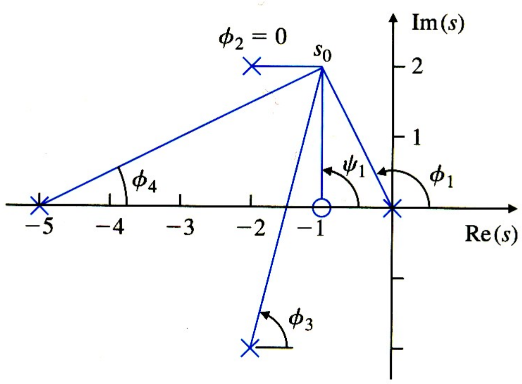

Figure 5.5: Measuring the phases of Example 5.2.

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Example 5.2 demonstrates measuring phase is easy, however measuring phase at every point in the s-plane is not practical. Therefore, we need some general rules for determining where the root locus is.

5.2 Rules & steps for plotting the root-locus

The following transfer function is used for illustrating the steps for plotting the root-locus

$$G(s) = \frac{1}{s[(s+4)^2+16]}$$

Rule 1. The root-locus is symmetric with respect to the real axis.

This follows from the assumption the $G(s)$ is a ratio of two polynomials with real coefficients. So, the characteristic polynomial roots are either real or occur in complex conjugate pair.

Step 1. Draw the axes of the s-plane to a suitable scale and enter $\times$ for each pole of $G(s)$ and a $\bigcirc$ for each zero. See Figure 5.6.

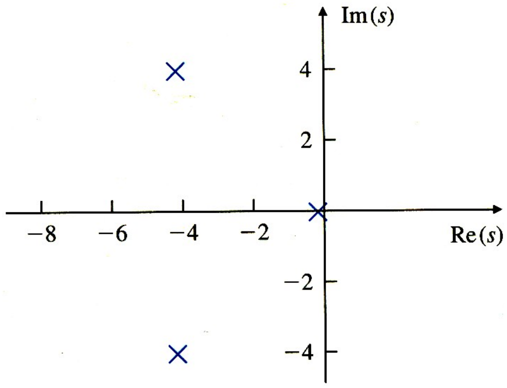

Figure 5.6: Step 1: Mark the poles and zeros.

Rule 2. The root-locus includes all points on the real axis to the left of an odd number of poles and zeros.

This follows from the angle criterion, we consider first that all poles and zeros of the open-loop transfer function are on the real axis, and we test points on the real axis to determine if these points are on the locus.

Consider an open-loop transfer function $G(s)$ of two poles and one zero as illustrated in Figure 5.7. If we take a test point s on the real axis to the right of the zero z_1 as shown in Figure 5.7(a) we find that $\angle G(s) = 0$. Hence, the angle criterion is not satisfied, and we can see that any point to the right of the zero z_1 cannot be on the root locus.

Consider now a point s between the zero z_1 and the pole p_1. In this case

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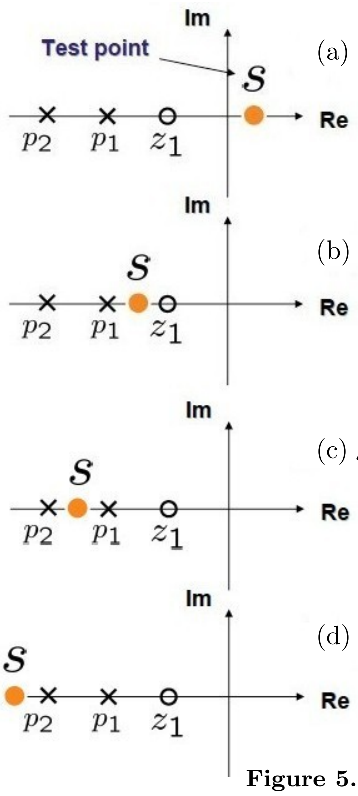

(a) $\angle G(s) = \underbrace{\angle s - z_1}_{0} - \underbrace{\angle s - p_1}_{0} - \underbrace{\angle s - p_2}_{0}$

Angle criterion is not satisfied!

(b) $\angle G(s) = \underbrace{\angle s - z_1}_{180} - \underbrace{\angle s - p_1}_{0} - \underbrace{\angle s - p_2}_{0}$

Angle criterion is satisfied!

(c) $\angle G(s) = \underbrace{\angle s - z_1}_{180} - \underbrace{\angle s - p_1}_{180} - \underbrace{\angle s - p_2}_{0}$

Angle criterion is not satisfied!

(d) $\angle G(s) = \underbrace{\angle s - z_1}_{180} - \underbrace{\angle s - p_1}_{180} - \underbrace{\angle s - p_2}_{180}$

Angle criterion is satisfied!

Figure 5.7: Real axis locus.

$\angle G(s) = 180°$, as shown in Figure 5.7(b). However the angles from the poles p_1 and p_2 are still 0°. Thus the angle requirement is satisfied, and any point between z_1 and p_1 is on the locus.

For a point s between p_1 and p_2, (see Figure 5.7(c)) the angle from z_1 is still 180°, as now is the angle from p_1. The angle from p_2 is still 0°; hence the angle requirement is not satisfied and no points between p_1 and p_2 are on the locus. If the point s is to the left of the pole p_2, the angles from z_1, p_1, and p_2 are all 180°, and the angle criterion is satisfied as shown in Figure 5.7(d).

For the case that we have complex poles or zeros, the preceding discussion still applies. For example, two complex conjugate poles are shown in Figure 5.8. Since complex poles (and zeros) must occur in conjugate pairs, the sum of the angles from a pair of poles (or zeros) to a point on the real axis will always be 0° (or 360°). Hence complex poles and zeros do not affect the part of the root locus that lies on the real axis.

Summary:

- The angle contribution from a pole or zero to the left of s is 0°.
- The angle contribution from a pole or zero to the right of s is 180°.
- The angle contribution from a pair of complex conjugate poles or zeros cancels out.

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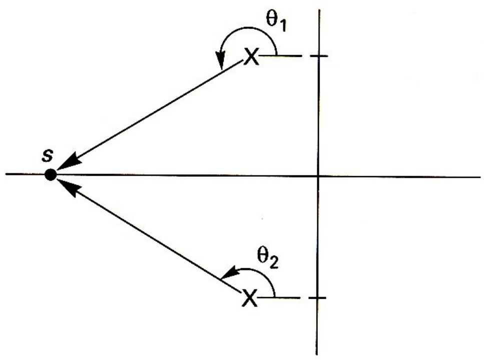

Figure 5.8: Real axis locus.

STEP 2. Find the real axis portion of the locus.

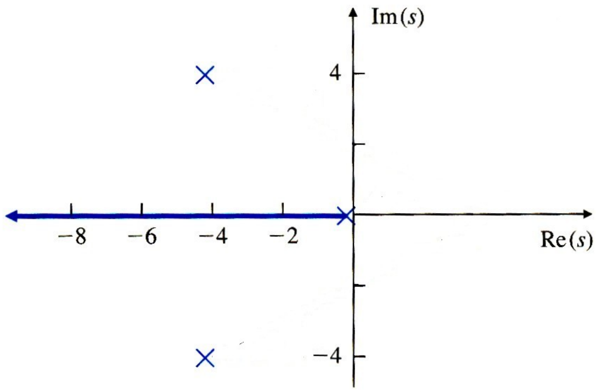

Figure 5.9: The real axis parts of the locus are to the left of an odd number of poles and zeros.

RULE 3. The root-locus originate on the open-loop poles for $K = 0$ and terminate at the open-loop zeros when they exist, otherwise it terminates at infinity.

Suppose that

$$G(s) = \frac{n(s)}{d(s)} = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \quad \text{where} \quad n > m$$

$$= \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_a s^{n-1} + \cdots + a_n}$$

We assume that $n(s)$ and $d(s)$ are monic polynomials (monic means the coefficient of the highest power of s is 1). The closed-loop characteristic equation

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can be written as

$$1 + K\frac{n(s)}{d(s)} = 0 \Longleftrightarrow d(s) + Kn(s) = 0 \Longleftrightarrow \frac{1}{K} + \frac{n(s)}{d(s)} = 0 \qquad (5.7)$$

If $K = 0$, then from (5.7) implies $d(s) = 0$ (i.e., the poles of $G(s)$). Therefore, for $K = 0$ the roots of the closed-loop characteristic equation $1+KG(s) = 0$ are the open-loop poles. *The points of the root-locus where $K = 0$ are sometimes called the starting or departure points of the root-locus.*

As k approaches infinity but s remains finite (the case $s \to \infty$ will be discussed in Rule 4), (5.7) implies $n(s)/d(s) = 0$ which in turn implies $n(s) = 0$. Therefore, as K approaches infinity the roots of $1 + KG(s)$ are the open-loop zeros. *The points of the root-locus where $K = \infty$ are sometimes called the ending or arrival points of the root-locus.*

Note from the above discussion that we have n poles and m zeros. If m of the n poles will terminate at m zeros, where will the $n - m$ poles terminate. As Rule 3 states they will terminate at infinity, the question remains which infinity, Rule 4 next clarifies the matter.

RULE 4. If $G(s)$ has α zeros at infinity, the root-locus will approach α asymptotes as $K \to \infty$. The angles of asymptotes are

$$\phi_A = \pm\frac{(2l+1)}{\alpha}\pi \qquad l = 0, 1, 2, \cdots$$

The asymptotes intersect the real axis at

$$\sigma_A = \frac{\sum \textbf{poles} - \sum \textbf{zeros}}{\textbf{number of poles} - \textbf{number of zeros}}$$

Recall from the discussion of Rule 3 for $K \to \infty$, $G(s) = 0$ if $n(s)$ is zero for a finite s. The root locus will approach the open-loop zeros. To see a second manner in which $G(s)$ may go to zero, we express the characteristic equation $1 + KG(s) = 0$ as

$$1 + K\frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_a s^{n-1} + \cdots + a_n} = 0 \qquad (5.8)$$

Since $n > m$, it is clear that $G(s)$ goes to zero as $s \to \infty$. In fact, for very large values of s (5.8) can be approximated by

$$1 + K\frac{1}{(s - \sigma_A)^{n-m}} = 0 \qquad (5.9)$$

To see why (5.9) is a good approximation to (5.8), try to imagine what would we see if we could observe the locations of poles and zeros from a distance point near infinity: They would appear to cluster near the s-plane origin as shown in Figure 5.10(a). Thus m zeros would cancel the effects of m poles, and the other $n - m$ poles would appear to be in the same place, namely at $s = \sigma_A$ as shown in Figure 5.10(b). If $\alpha = n - m$ we may write (5.9) as

$$1 + K\frac{1}{(s - \sigma_A)^{\alpha}} = 0$$

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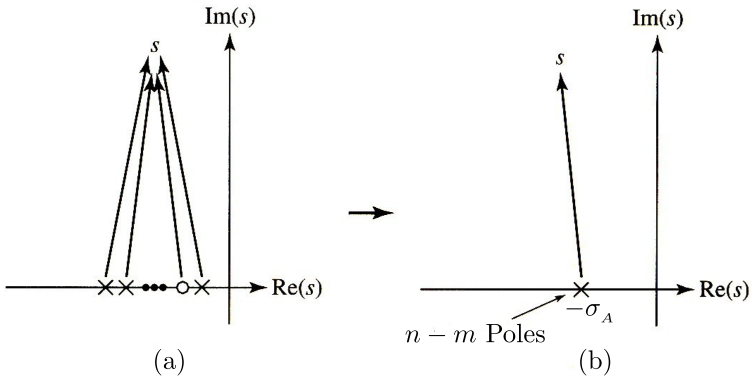

Figure 5.10: Determination of angles of asymptotes.

We say that the locus of (5.8) is asymptotic to the locus of (5.9) for large values of K and s.

To find the locus, no matter how far away the point s is on the s-plane it must satisfy the angle criterion. Since all α poles appear to be in the same place the angle condition gives

$$\alpha\phi_A = \pm r(180^\circ) \qquad r = 1, 3, 5, \cdots$$
$$\Longrightarrow \phi_A = \pm r\frac{(180^\circ)}{\alpha}$$

The angles ϕ_A are the angles of asymptotes of the root-locus. Table 5.2 gives these angles for small values of α.

Table 5.2: Angles of asymptotes.

α	*Angles*
0	No asymptotes
1	180°
2	±90°
3	±60°, 180°
4	±45°, ±135°
5	±36°, ±108°, 180°

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FINDING σ_A

To determine σ_A we make use of polynomial properties discussed in Section 4.3.1. Write $G(s)$ as

$$G(s) = \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

$$= \frac{s^m - \left(\sum_{i=1}^{m} z_i\right) s^{m-1} + \cdots}{s^n - \left(\sum_{i=1}^{n} p_i\right) s^{n-1} + \cdots}$$

Dividing both the numerator and the denominator by the numerator gives

$$G(s) = \frac{1}{s^{n-m} - \left(\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i\right) s^{n-m-1} + \cdots} \tag{5.10}$$

For very large values of s, $G(s)$ was approximated by

$$\frac{1}{(s - \sigma_A)^{n-m}} \tag{5.11}$$

The polynomial $(s - \sigma_A)^{n-m}$ in (5.11) can be written as

$$s^{n-m} + a_{n-1}s^{n-m-1} + a_{n-2}s^{n-m-2} + \cdots$$

where

$$a_{n-1} = -\sum_{i=1}^{n-m} p_i = -(n-m)\sigma_A$$

Hence (5.11) may be written as

$$\frac{1}{s^{n-m} - (n-m)\sigma_A s^{n-m-1} + \cdots} \tag{5.12}$$

Comparing (5.12) and (5.10) to order s^{n-m-1} yields

$$(n-m)\sigma_A = \sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i$$

Hence,

$$\sigma_A = \frac{\sum p_i - \sum z_i}{n-m}$$

Notice that in the sum $\sum p_i$ and $\sum z_i$ the imaginary parts always add to zero since complex poles and zeros always occur in complex conjugate pairs.

In summary the loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A. when the number of m finite zeros is less than the n number of poles, then $n - m$ sections of loci must end at zeros at infinity. these sections of loci proceed to the zeros at infinity along asymptotes as k approaches infinity. These linear asymptotes are centered at the point σ_A on the real axis.

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STEP 3. Draw the asymptotes for large values of K.

For our example,

$$\sigma_A = \frac{-4-4+0}{3-0} = -\frac{8}{3} = -2.67$$

and $\alpha = 3 \implies$ asymptotes at $\pm 60°$ and an asymptote at $180°$. The asymptotes at $\pm 60°$ are shown dashed in Figure 5.11. Notice that they cross the imaginary axis at $\pm 4.62j$. The asymptote at $180°$ was already found in Step 2.

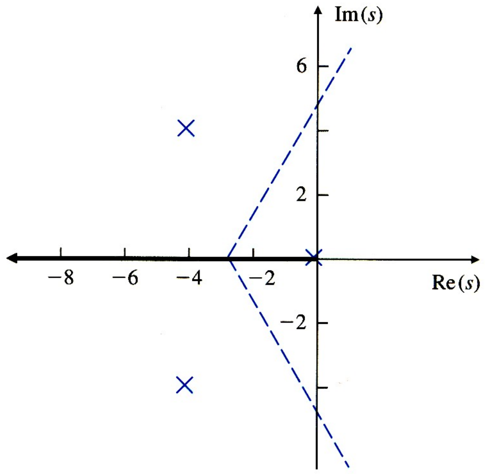

Figure 5.11: Draw the asymptotes.

RULE 5. The root-locus departs from a complex pole p_j at an angle θ_d given by

$$\theta_d = \sum_i \theta_{zi} - \sum_{i \neq j} \theta_{pi} \pm r(180°) \qquad r = 1, 3, 5, \cdots$$

The root-locus arrives at a complex zero z_j at an angle θ_a given by

$$\theta_a = \sum_i \theta_{pi} - \sum_{i \neq j} \theta_{zi} \pm r(180°) \qquad r = 1, 3, 5, \cdots$$

The most important use of this rule is to compute the angle of departure from a complex pole. This angle of departure can sometimes be an aid in determining the final shape of the root locus. To illustrate this rule consider the poles and the zero shown in Figure 5.12. The vector angles at one complex pole p_1 is also shown in Figure 5.12. The radius of the circle around the pole p_1 is actually very small in relation to the distance to the zero and the other pole. The angles at a test point s_0, an infinitesimal distance from p_1, must meet the angle criterion. Therefore,

$$\alpha - \theta_1 - \theta_2 = 180°$$

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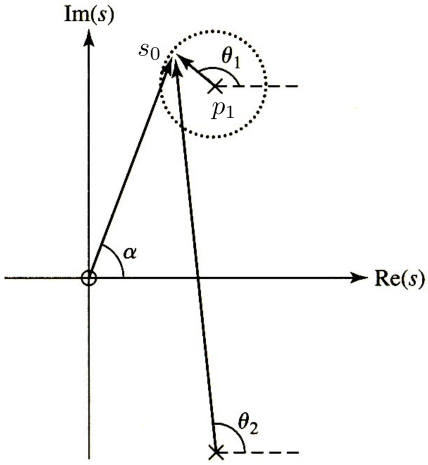

Figure 5.12: Determining the angle of departure.

and the angle of departure at pole p_1 is

$$\theta_1 = \alpha \pm 180° - \theta_2$$

Note that we select either $+180°$ or $-180°$ above to ensure that θ_1 is always selected to be in the range $-180° < \theta_1 < +180°$.

STEP 4. Compute the departure and arrival angles.

First we take a test point s_0 very near pole 2 at $-4 + 4j$ and compute the angle of $G(s_0)$. This situation is sketched in Figure 5.13. We select the test

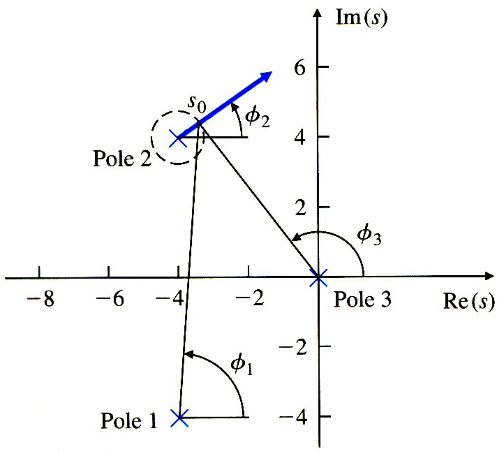

Figure 5.13: Compute the angle of departure.

point close enough to pole 2 that the angles ϕ_1 and ϕ_3 to the test point can be

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considered the same as those angles to pole 2. Thus $\phi_1 = 90°$ and $\phi_3 = 135°$, and ϕ_2 can be calculated from the angle condition

$$-90° - \phi_2 - 135° = \pm 180°$$

To ensure $-180° < \phi_2 < +180°$ we have

$$-\phi_2 = 90° + 135° - 180°$$
$$\phi_2 = -45°$$

as shown in Figure 5.14. By the complex conjugate symmetry of the plots, the angle of departure of the locus near pole 1 at $-4 - 4j$ will be $+45°$. **Note** *for a multiple pole of order q we must count the angle from the pole q times.*

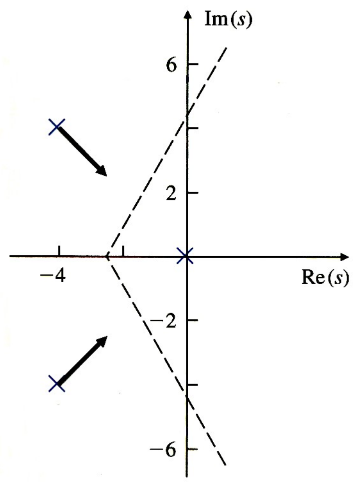

Figure 5.14: Actual angle of departure.

STEP 5. Estimate (or compute) the points where the locus crosses the imaginary axis.

The points where the root-locus intersect the imaginary axis of the s-plane, and the corresponding values of K, may be determined by means of the Routh-Hurwitz criterion. For the third-order example we are using, the characteristic equation is

$$1 + \frac{K}{s[(s+4)^2 + 16]} = 0$$

which is equivalent to

$$s^3 + 8s^2 + 32s + K = 0$$

The Routh array for this polynomial is

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s^3	1	32
s^2	8	K
s^1	$\frac{256-K}{8}$	0
1	K	

In this case we see that the s^1 row coefficients are all zeros when $K = 256$ indicating a root on the imaginary axis. Thus $K = 256$ must correspond to a solution at $s = j\omega_0$ for some ω_0. Substituting this data into the auxiliary equation gives us

$$-8\omega_0^2 + 256 = 0$$

Clearly the solution is $\omega_0 = \pm\sqrt{32} = \pm 5.66$, which is plotted in Figure 5.15.

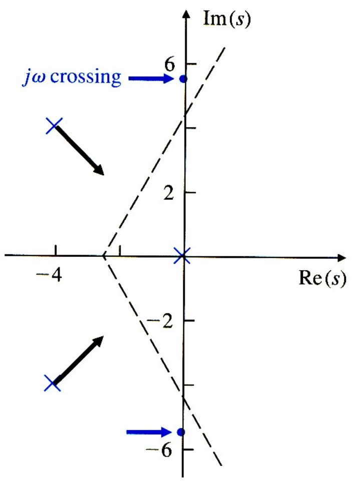

Figure 5.15: Find the imaginary axis crossing points.

RULE 5. The breakaway points σ_b are among the real roots of

$$\frac{dG(s)}{ds} = 0$$

or, equivalently,

$$n(s)\frac{d}{ds}[d(s)] - d(s)\frac{d}{ds}[n(s)] = 0$$

where $n(s)$ and $d(s)$ are the numerator and denominator polynomials, respectively, of $G(s)$. The breakaway points are points at which two (or more) branches of the root locus leave the real axis. The example in Figure 5.2 provides an illustration of a breakaway point. In this case, there is a breakaway point at $s = -1$. Figure 5.16 shows both the root-locus and a plot of K as a function of real values of s between 0 and -2. The maximum occurs at $s = -1$ for $K = 1$. At the point where $K = 1$ the characteristic equation has a double root at $s = -1$. This is the maximum gain for which the poles are real; higher gains result in

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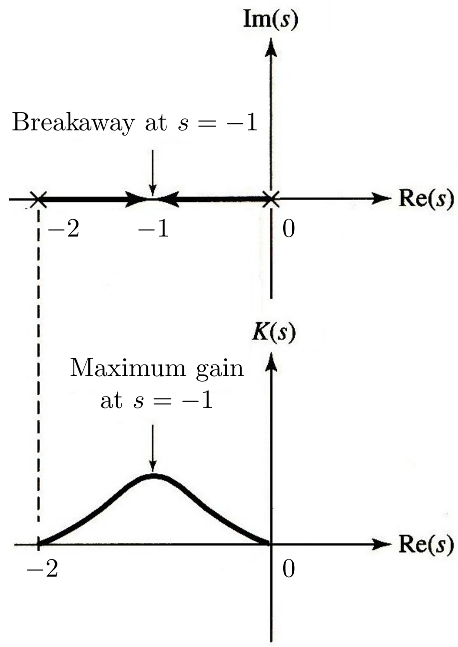

Figure 5.16: Gain K as a function of s along the real axis.

complex roots. Notice in Figure 5.16 that the gain K, as a function of the real roots s, must have a local maximum at the breakaway points, so that, with

$$K = -\frac{1}{G(s)}$$

and s considered a real variable, we require

$$\frac{dK}{ds} = 0 \tag{5.13}$$

If we express $G(s)$ as a ratio of two polynomials $n(s)$ and $d(s)$ the above equation can be written as

$$\frac{dK}{ds} = \frac{d}{ds}\left(-\frac{1}{G(s)}\right) = \frac{d}{ds}\left(-\frac{d(s)}{n(s)}\right) = 0$$

The differentiation with respect to s, yields

$$\frac{d}{ds}\left(-\frac{d(s)}{n(s)}\right) = -\left[d(s)(-1)\frac{1}{n^2(s)}\frac{d}{ds}[n(s)] + \frac{1}{n(s)}\frac{d}{ds}[d(s)]\right]$$

Equating the right hand side of the equation above to zero implies

$$n(s)\frac{d}{ds}[d(s)] - d(s)\frac{d}{ds}[n(s)] = 0$$

It is important to point out that the condition for a breakaway point given in (5.13) is *necessary* but *not sufficient.* In other words, all breakaway points on

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the root-locus must satisfy (5.13), but not all solutions of (5.13) are breakaway points.

STEP 6. Determine the breakaway point σ_b.

In our example, $G(s)$ is

$$G(s) = \frac{1}{s[(s+4)^2+16]}$$

and we have

$$n(s) = 1 \qquad \frac{d\,n(s)}{ds} = 0$$

$$d(s) = s^3 + 8s^2 + 32s \qquad \frac{d\,d(s)}{ds} = 3s^2 + 16s + 32$$

the points of possible multiple roots or breakaway are given by

$$3s^2 + 16s + 32 = 0$$

or

$$s_0 = -2.67 \pm 1.89j$$

The breakaway point must be real and lies on the root-locus, hence, for this example there is no breakaway point.

STEP 7. Complete the sketch.

The complete locus for our third-order example is drawn in Figure 5.17. It combines all the results found so far, that is, the real axis segment, the number of asymptotes and their angles, the angles of departure from the poles, and the imaginary axis crossing points.

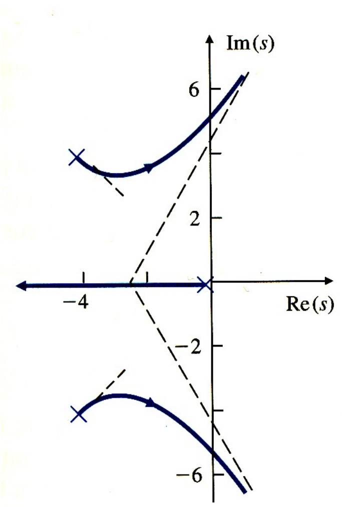

Figure 5.17: Complete the sketch.

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5.3 Examples

In all the examples to follow, $G(s)$ is the transfer function of a system to be controlled using constant-gain in the forward path, with $K \geq 0$.

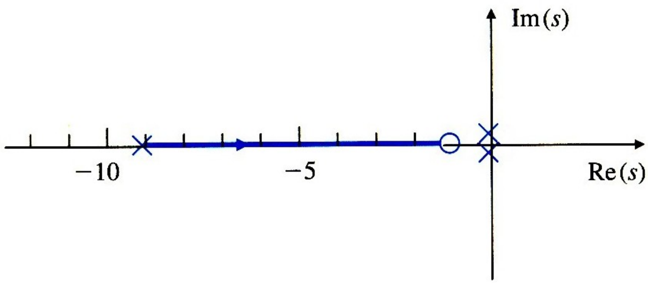
Example 5.3

Find the root locus for

$$G(s) = \frac{s+1}{s^2(s+9)}$$

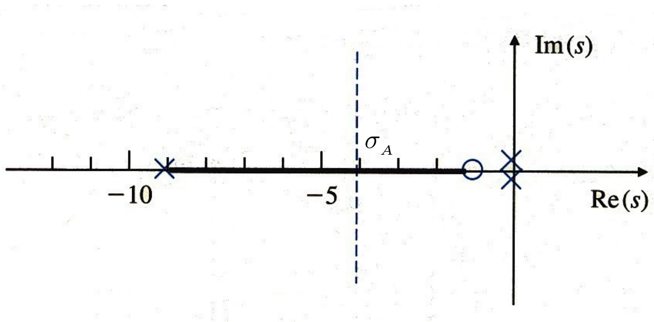
■ **Solution**

STEPS 1 and 2. Mark the poles and zeros on the s-palne and draw the real axis portion of the locus:



STEP 3. Draw the $n - m = 3 - 1 = 2$ asymptotes:

$$\phi_A = \pm r\frac{(180^\circ)}{n-m} = \pm\frac{(180^\circ)}{2} = \pm 90^\circ \qquad \sigma_A = \frac{\sum p_i - \sum z_1}{n-m} = \frac{-9-0-(-1)}{3-1} = -4$$



STEP 4. We compute the departure angles from the poles. We draw a small circle around the two poles at $s = 0$. The angles from the zero at -1 and from the pole at -10 are both zero, and the angles from the two poles at the origin are the same. Therefore, the root locus condition is

$$-2\theta_d = r180^\circ = \pm 90^\circ$$

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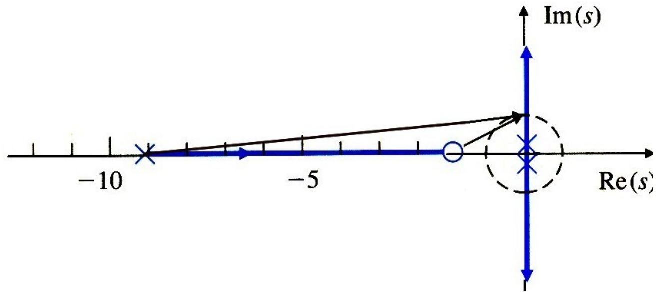

STEP 5. We compute the points where the locus crosses the imaginary axis:

$$1 + K\frac{s+1}{s^2(s+9)} = 0$$

$$s^3 + 9s^2 + Ks + K = 0$$

The Routh array for this polynomial is

s^3	1	K
s^2	9	K
s^1	$\frac{9K-K}{9}$	0
1	K	

The entries in the first column are all positive if $K > 0$, so the equation has no roots in the RHP for positive values of K.

STEP 6. We locate the points of multiple roots, which will include breakaway and break-in points:

$$n(s) = s + 1 \qquad \frac{d\,n(s)}{ds} = 1$$

$$d(s) = s^3 + 9s^2 \qquad \frac{d\,d(s)}{ds} = 3s^2 + 18s$$

The possible multiple roots are at

$$(s+1)(3s^2 + 18s) - (s^3 + 9s^2)(1) = 0$$
$$2s^3 + 12s^2 + 18s = 0$$
$$s = 0, -3, -3$$

The points of multiple roots are on the locus, but we have repeated roots in the derivative, which indicates that we have three roots at the same place. Note we can apply the rule of departure angles to the triple root at $s = -3$, we find that

$$180^\circ - 3\theta_d = r180^\circ$$
$$\theta_d = 0^\circ, \pm 120^\circ$$

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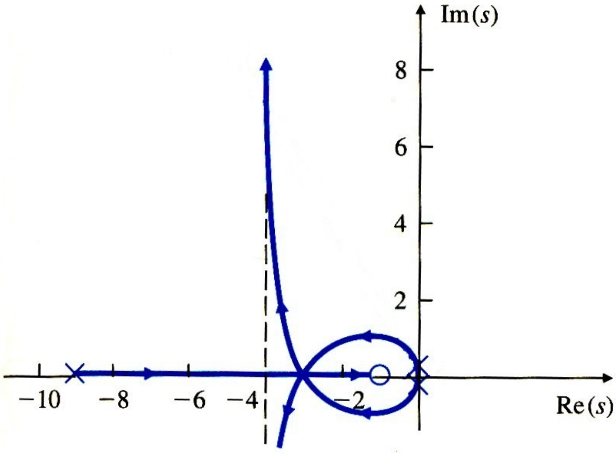


Figure 5.18: Root locus for $G(s) = \dfrac{(s+1)}{s^2(s+9)}$.

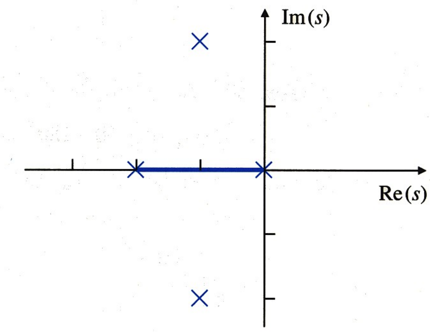
STEP 7. The complete sketch is given in Figure 5.18. ■

Example 5.4 Find the root locus for

$$G(s) = \frac{1}{s(s+2)[(s+1)^2+4]}$$

■ **Solution**

STEPS 1 and 2. Mark the poles and zeros on the s-palne and draw the real axis portion of the locus:



STEP 3. Draw the asymptotes:

$$\phi_A = \pm r\frac{(180°)}{4-0} \qquad \sigma_A = \frac{-2-1-1-0+0}{4-0}$$

$$= \pm 45°, \pm 135° \qquad = -1$$

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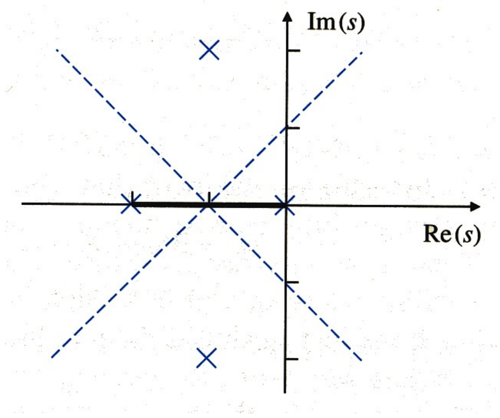

STEP 4. The departure angle at the complex pole at $-1 + 2j$ is

$$0 - 116.6^\circ - 63.4^\circ - 90^\circ - \theta_d = \pm 180^\circ$$
$$\theta_d = -90^\circ$$

We can observe at once that, along the line $s = -1 + j\omega$ the angle criterion is always satisfied. This is a special case since the angles of the real axis poles to any point on this line will form an isosceles triangle and always add to 180°.

STEP 5. We compute the crossings of the imaginary axis. The characteristic equation is

$$s^4 + 4s^3 + 9s^2 + 10s + K = 0$$

The Routh array for this polynomial is

$$\begin{array}{c|ccc} s^4 & 1 & 9 & K \\ s^3 & 4 & 10 & \\ s^2 & 6.5 & K & \\ s^1 & \dfrac{65-4K}{6.5} & 0 & \\ 1 & K & & \end{array}$$

In this case we see that the s^1 row coefficients are all zeros when $K = 16.25$ indicating a root on the imaginary axis. Thus $K = 16.25$ must correspond to a solution at $s = j\omega_0$ for some ω_0. Substituting this data into the auxiliary equation gives us

$$-6.5\omega_0^2 + 16.25 = 0$$

Clearly the solution is $\omega_0 = \pm\sqrt{2.5} = \pm 1.58$.

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STEP 6. We locate possible multiple roots:

$$n(s) = 1 \qquad \frac{d\,n(s)}{ds} = 0$$

$$d(s) = s^4 + 4s^3 + 9s^2 + 10s \qquad \frac{d\,d(s)}{ds} = 4s^3 + 12s^2 + 18s + 10$$

The possible multiple roots are at

$$4s^3 + 12s^2 + 18s + 10 = 0$$

From step 4 we notice that the line at $s = -1 + j\omega$ is on the locus, so there must be a breakaway point at $s = -1$, which can be divided out. That is , we can easily show that

$$4s^3 + 12s^2 + 18s + 10 = (s+1)(4s^2 + 8s + 10) = 0$$

The quadratic has roots $-1 \pm 1.22j$. Since these points are on the line between the complex poles, they are points of multiple roots on the locus.

STEP 7. The complete sketch is given in Figure 5.19. Notice that we have complex multiple roots. Branches of the locus come together at $-1 \pm 1.22j$ and break away at 0° and 180°. ■

Sketching root loci relies heavily on experience. Figure 5.20 gives several loci for low-order systems; these should be studied to familiarize yourself with some of the characteristics of root loci.

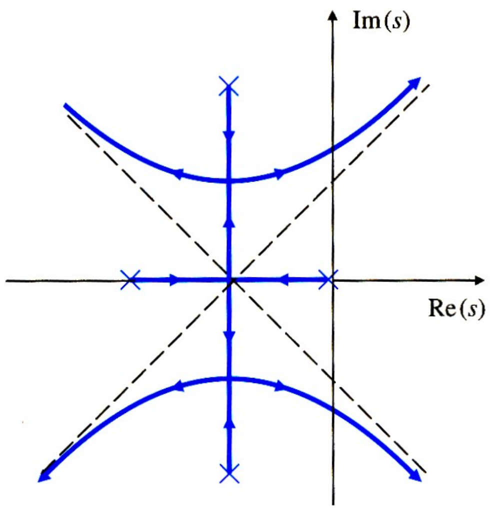

Figure 5.19: Root locus for $G(s) = \dfrac{1}{s(s+2)[(s+1)^2+4]}$.

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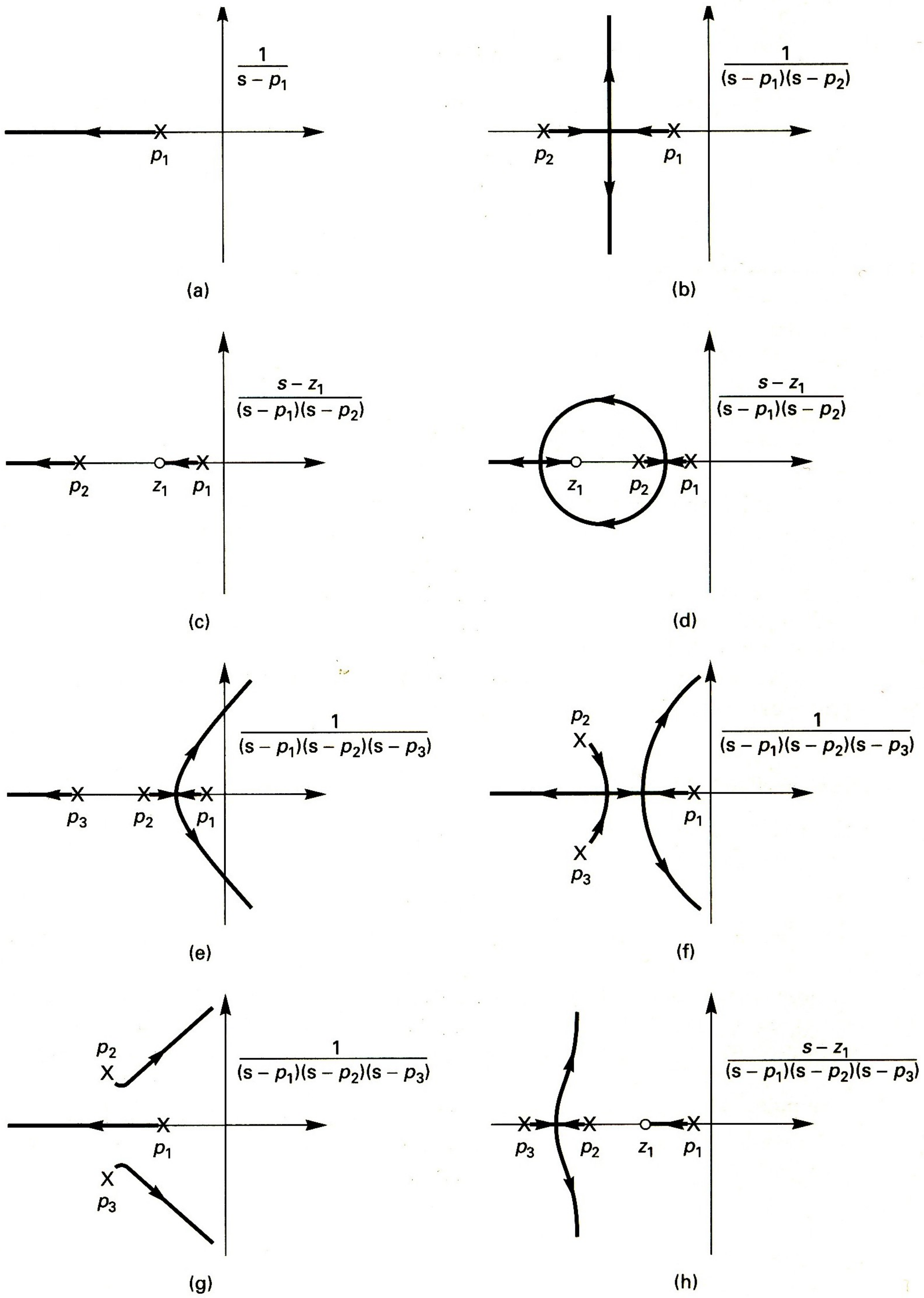

Figure 5.20: Loci for low-order systems.

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5.4 The complementary root locus

We now consider modifying the root-locus procedure to permit analysis of negative values of K. Following are the rules for plotting a complementary root-locus:

1. The root-locus is symmetric with respect to the real axis.
2. The root-locus includes all points on the real-axis to the left of an even number of poles and zeros.
3. The root-locus originates on the poles (for $K = 0$) and terminate at the zeros (as $K \to -\infty$) of $G(s)$.
4. If $G(s)$ has α zeros at infinity, the root-locus will approach α asymptotes as K approach $-\infty$. The angles of asymptotes are

$$\phi_A = \pm \frac{2l\pi}{\alpha} \qquad l = 0, 1, 2, \cdots$$

   and the asymptotes intersect the real axis at

$$\sigma_A = \frac{\sum p_i - \sum z_i}{\alpha}$$

5. The angle of departure from a pole p_j is given by

$$\theta_d = \sum_i \theta_{zi} - \sum_{i \neq j} \theta_{pi} + 2l\pi$$

   and arrives at a zero z_j at an angle

$$\theta_a = \sum_i \theta_{pi} - \sum_{i \neq j} \theta_{zi} + 2l\pi$$

6. The breakaway points σ_b are roots of

$$\frac{dG(s)}{ds} = 0$$

Example 5.5 Consider

$$G(s) = \frac{s}{(s - 0.5 - 2j)(s - 0.5 + 2j)}$$

■ **Solution** The root locus and the complementary root locus are shown in Figure 5.21. In the complementary root-locus, the locus on the real axis occurs to the left of an even count of poles and zeros. Since zero is considered even, root locus on the real axis will occur only to the right of the zero at the origin. The break-in points are $s = \pm 2.06$. Thus the break-in point for the complementary root locus is at $s = 2.06$. After the break-in, one closed-loop pole migrates to the zero at the origin and the other to the right toward infinity.

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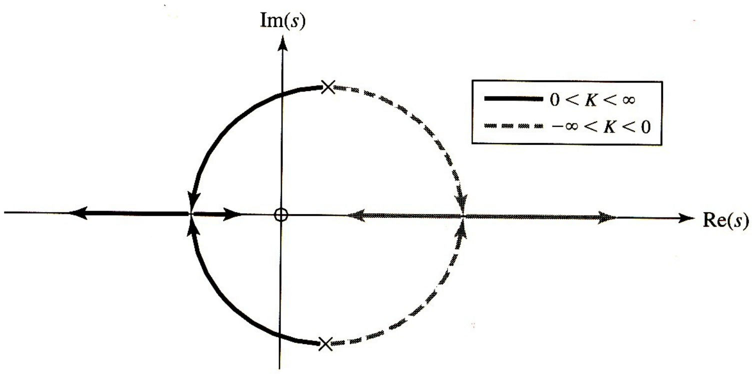

Figure 5.21: Root locus for $G(s) = \dfrac{s}{(s - 0.5 - 2j)(s - 0.5 + 2j)}, -\infty < K < \infty.$

Example 5.6

Consider

$$G(s) = \frac{s + 1}{s(s + 4)(s + 10)}$$

■ **Solution** The complementary root locus on the real axis occurs to the right of the pole at the origin, between the zero at $s = -1$ and the pole at $s = -4$, and to the left of the pole at $s = -10$. The number of zeros at infinity, $\alpha = 2$, so there are two asymptotes at 0° and 180°. The root locus and the complementary root locus are shown in Figure 5.22.

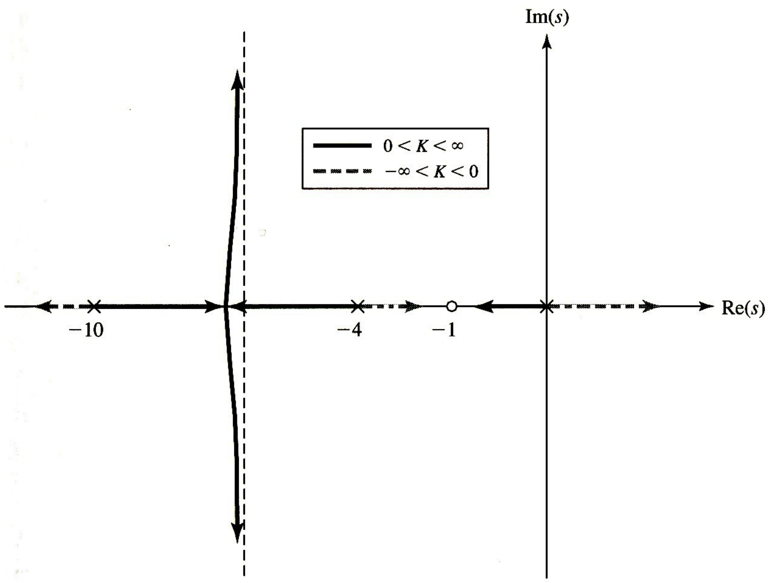

Figure 5.22: Root locus for $G(s) = \dfrac{s + 1}{s(s + 4)(s + 10)}, -\infty < K < \infty.$

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5.5 Root locus design

The root-locus is a plot of all possible locations for roots to the equation $1 + KG(s) = 0$ for some real positive value of K. The purpose of design is to select a particular value of K that will meet the design specifications. Consider for example the locus of

$$G(s) = \frac{1}{s[(s+4)^2 + 16]}$$

For this transfer function, the locus was plotted in Figure 5.17 and is repeated here in Figure 5.23. On Figure 5.23 the lines corresponding to a damping ratio of $\zeta = 0.5$ are sketched, and the points where the locus crosses these lines are marked with ($\bullet$). Suppose we wish to set the gain so that the poles are located at the dots. This corresponds to selecting the gain so that two of the closed-loop poles have a damping ratio of $\zeta = 0.5$. What is the value of K when a root is

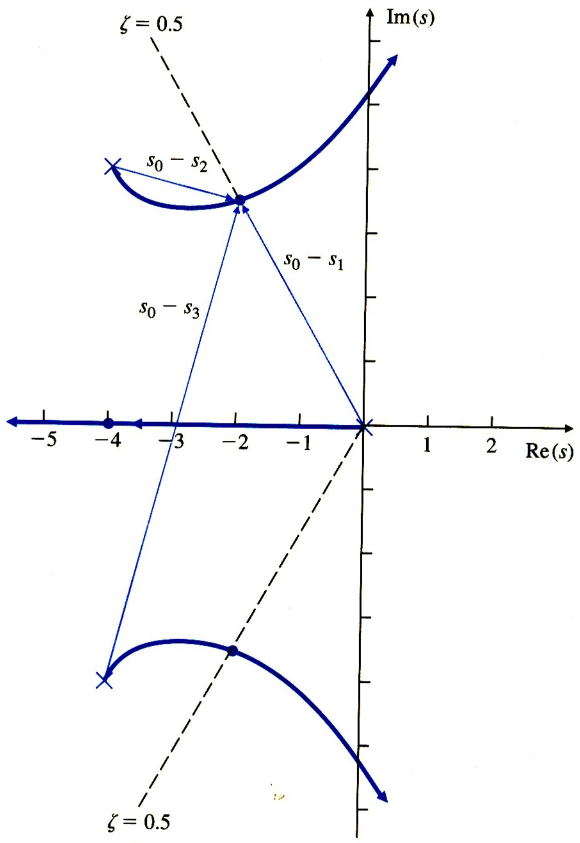

Figure 5.23: Root locus for $G(s) = \dfrac{1}{s[(s+4)^2 + 16]}$ showing calculations of gain K.

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at the dot? The value of K is given by

$$K = \frac{1}{|G(s_0)|}$$

where s_0 is the coordinate of the dot. On the figure we have plotted three vectors marked $s_0 - s_1$, $s_0 - s_1$, and $s_0 - s_3$, which are the vectors from the poles of $G(s)$ to the point s_0. (Since $s_1 = 0$, the first vector equals s_0.) Therefore,

$$K = \frac{1}{|G(s_0)|} = |s_0||s_0 - s_2||s_0 - s_3|$$

By measuring the lengths of these vectors and multiplying the lengths together, provided that the scale of the imaginary and real axes is identical, we can compute the gain to place the roots at the dot ($s = s_0$). Using the scale of the figure we estimate that

$$\begin{aligned} |s_0| &\approx 4 \\ |s_0 - s_2| &\approx 2.1 \\ |s_0 - s_3| &\approx 7.7 \end{aligned}$$

Thus the gain is estimated to be

$$K = 4(2.1)(7.7) \approx 65$$

We conclude that if K is set to the value 65, then a root of $1 + KG(s)$ will be at s_0, which has a damping ratio of 0.5. Another root is at the conjugate of s_0. Where is the third root? The third root lies on the branch of the locus along the negative real axis. Usually we take a test point, compute a trial gain, and repeat this process until we found the point where $K = 65$. However, in this case we can make use of polynomial properties that the open-loop and closed-loop sum is the same if $m < n - 1$. If $G(s)$ has at least two more poles than zeros, we have

$$\sum \text{open-loop poles} = \sum \text{closed-loop poles} \tag{5.14}$$

From Figure 5.23 we estimate that $s_0 = -2 + 3.5j$. Since the starting point was at $s = -4 + 4j$, the root has moved approximately two units to the right. The conjugate has moved an equal distance. The third root must be moved far enough to the left to keep the sum in (5.14) fixed, so the third root must have moved $2 + 2$ units to the left of where it began at $s = 0$. We have marked the new location at -4 with the third dot. Considering this point as a test point one can check if the gain at this point is $K = 65$.

If the closed-loop dynamic response as determined by the root locations is satisfactory, then the design can be completed by gain selection alone. However if no value of K satisfies all the constraints, as is typically the case, then additional modifications are necessary to meet the system specifications.

5.6 Dynamic compensation

If the plant dynamics are of such a nature that a satisfactory design cannot be obtained by a gain adjustment alone, then some modification or compensation of the plant dynamics are needed. A variety of compensation techniques are

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available, only two such techniques that have been found simple and effective will be discussed here. These are lead and lag compensation. **Lead compensation** acts mainly to speed up a response by lowering rise time and decreasing the transient overshoot. **Lag compensation** is usually used to improve steady-state accuracy of the system.

Compensation with a transfer function of the form

$$D(s) = K\frac{s+z}{s+p}$$

is called lead compensation if $|z| < |p|$ and lag compensation if $|z| > |p|$. Compensation is typically placed in series with the plant in the feedforward path, as shown in Figure 5.24.

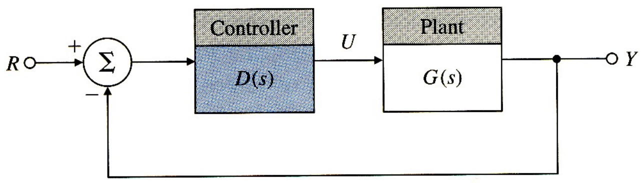

Figure 5.24: Feedback system with compensation.

The characteristic equation of the system in Figure 5.24 is

$$1 + D(s)G(s) = 0$$
$$1 + KL(s) = 0$$

where K and $L(s)$ are selected to put the equation in root-locus form as before.

5.6.1 Lead compensation

To see the basic effect of lead compensation on a system, we first consider a simplified proportional control for which $D(s) = K$. If we apply this compensation to a second order system with transfer function

$$G(s) = \frac{1}{s(s+1)}$$

The root locus with respect to K is shown as the solid-line portion of the locus in Figure 5.25. Also shown in Figure 5.25 is the locus produced by proportional plus derivative control, where $D(s) = K(s+2)$. The modified locus is the circle sketched with dashed lines. Notice that the effect of the zero is to move the locus to the left, toward the more stable part of the s-plane.

The trouble with choosing $D(s)$ based on only a zero is that the physical realization would contain a differentiator that would greatly amplify the high frequency noise present from the sensor signal. To remedy this we simply add a high frequency pole, perhaps at $s = -20$ to give

$$D(s) = K\frac{s+2}{s+20}$$

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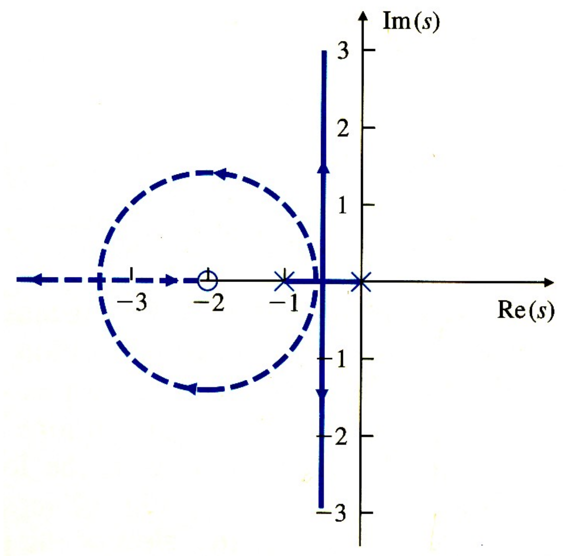

Figure 5.25: Root locus for $G(s) = \frac{1}{s(s+1)}$ without compensation (solid line), and with compensation $D(s) = s + 2$ (dashed lines).

The resulting transfer function is thus lead compensation. The root locus with such compensator is shown in Figure 5.26.

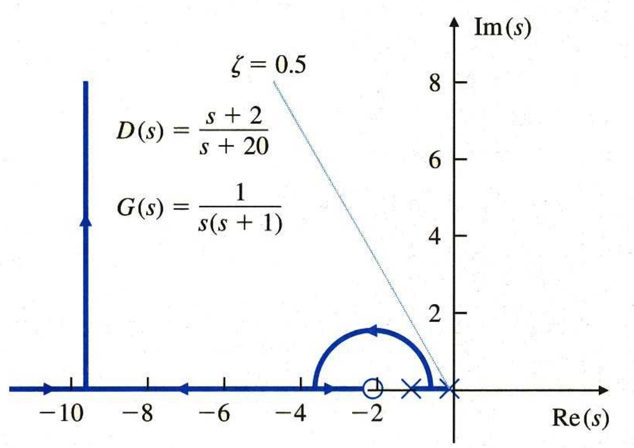

Figure 5.26: Root locus for $G(s) = \frac{1}{s(s+1)}$ with lead compensation $D(s) = \frac{s+2}{s+20}$.

To see the effect of the pole on the compensation consider moving the pole further to the right at $s = -10$, i.e, nearer to the zero. The root locus is shown in Figure 5.27. Notice the effect of moving the pole nearer to the zero, we are reducing the effect of the zero we placed earlier. In fact, we are moving back

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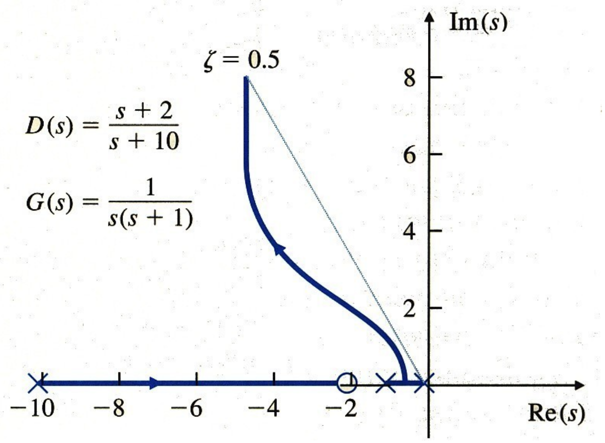

Figure 5.27: Root locus for $G(s) = \dfrac{1}{s(s+1)}$ with lead compensation $D(s) = \dfrac{s+2}{s+10}$.

to the uncompensated shape. If we move the pole too far to the left, the magnification of noise at the output of $D(s)$ is too great, since the differentiator will dominate the compensator. Therefore, the choice of pole location is a compromise between the conflicting effects of noise suppression and compensation effectivness.

Example 5.7

Find a compensation for

$$G(s) = \frac{1}{s(s+1)}$$

that will provide two closed-loop dominant poles having damping ratio $\zeta = 0.707$ and settling time of 2s. Furthermore, determine the value of the gain K to achieve this.

■ **Solution** The uncompensated root locus is shown in Figure 5.25. We need $T_s = 2$, which implies that $\zeta\omega_n = 2$. Hence, the first requirement is satisfied if we force the root locus to pass through the point $-2 + j2$ corresponding to $\omega_n = 2\sqrt{2}$ and $\zeta = 0.707$. Notice that we need to move the root locus to the left. This is achieved with a lead compensator of the form

$$D(s) = K\frac{s+z}{s+p}$$

Selecting values of z and p is done by trial and error. In general the compensator zero is placed in the neighborhood of the real part of closed-loop pole and the compensator pole is placed at a distance 3 to 20 times the value of the zero location.

The compensator pole position can now be determined by the angle criterion as shown in Figure 5.28

$$\alpha - (\theta_1 + \theta_2 + \beta) = \pm 180^\circ$$
$$90^\circ - (135^\circ + 116^\circ) \pm 180^\circ = \beta$$

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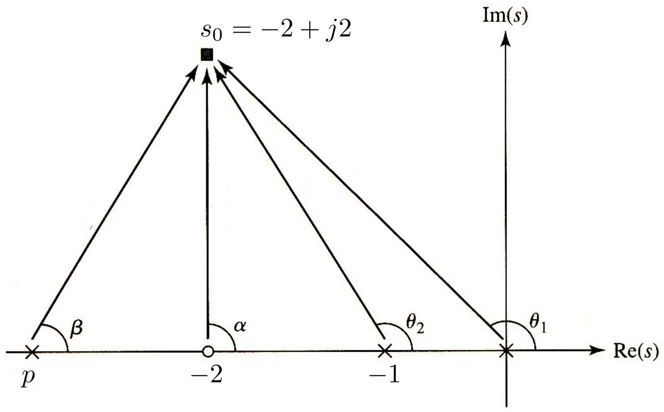

Figure 5.28: Angles for lead compensation.

Hence, $\beta = 19°$, and to determine the pole location p, we have

$$\tan(19°) = \frac{2}{|p-2|}$$

which implies $p = -7.8$. Finally, the value of the gain K can be determined using the magnitude criterion

$$\begin{aligned} K &= \frac{1}{|G(s_0)|} \\ &= \frac{|s_0||s_0+1||s_0+p|}{|s_0+z|} \\ &= \frac{\sqrt{8}\sqrt{5}\sqrt{37.64}}{2} \\ &\approx 19.4 \end{aligned}$$

The final design is then

$$D(s) = 19.4\frac{s+2}{s+7.8} \qquad \blacksquare$$

Although the design is complete and two of the closed-loop poles are already known, namely, the poles at $s = -2 \pm j2$. However the lead compensator introduces a third closed-loop pole. In this case the easiest way to locate this third pole is to make use of (5.14) since $m < n - 1$. Thus, the third closed-loop pole is at $s = -4.8$.

Example 5.8

Design a lead compensator for the system given by the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

that will provide a closed-loop damping ratio $\zeta = 0.5$ and natural frequency $\omega_n > 7$ rad/sec.

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■ **Solution** Since $\zeta = 0.5$ and $\omega_n = 7$, the closed loop poles are given by $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -3.5 \pm j6.062$. Let us choose $z = -3.5$. The compensator pole position can now be determined by the angle criterion as shown in Figure 5.29

$$90^\circ - 112.41^\circ - 120^\circ - \theta_p = -180^\circ$$

Hence the angle subtended by the compensator pole with the closed loop pole is 37.59°.

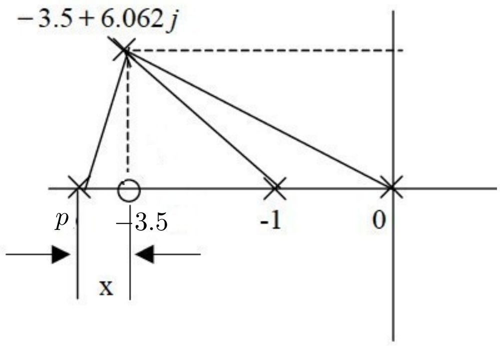

Figure 5.29: Angles for lead compensation.

To determine the pole location, we have

$$\tan(37.59^\circ) = \frac{2}{x}$$

which implies $p = -11.37$. Finally we use the magnitude criterion to calculate the gain $K \approx 75$. ■

5.6.2 Lag compensation

In this section we consider the design of lag compensators. As in the preceding sections, we assume that the compensator transfer function is first order and is given by

$$D(s) = K\frac{(s+z)}{(s+p)} \qquad |z| > |p|$$

It was shown that the effect of the addition of lead compensation is to shift the root locus to the left in the s-plane. However, lag compensators will tend to shift the root locus to the right in the s-plane, that is, toward the unstable region. Thus, in general, the shift to the right must be minimal to minimize the destabilizing effects. This small shift is assured by placing the pole and the zero of the compensator very close to each other. In fact we choose to place the compensator pole near the origin to approximate a perfect integrator. This should increase the system type by 1 that might be needed to improve steady-state error constants. The compensator zero is placed nearby so that the pole-zero pair does not significantly interfere with the dynamic response of the overall system. Thus, we want an expression for $D(s)$ that will yield a significant gain

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at $s = 0$ to raise an error constant and that is nearly unity (no effect) at the higher frequencies.

We now illustrate lag design with an example.

Example 5.9

Consider a system whose open loop transfer function is given by

$$G(s) = \frac{K}{s(s+2)}$$

Design a lag compensator so that the dominant poles of the closed loop system are located at $s = -1 \pm j$ and the steady state error to a unit ramp input is less than 0.2.

■ **Solution** For the specification that the steady state error of the system must not exceed 0.2, we have

$$\begin{aligned} K_v &= \lim_{s \to 0} sKD(s)G(s) \\ &= \lim_{s \to 0} sK\frac{(s+z)}{(s+p)}\frac{1}{s(s+2)} \\ &= \frac{Kz}{2p} \end{aligned}$$

We require the steady state error to be less than 0.2, i.e., $\frac{2p}{Kz} < 0.2$. Let us choose $p = 0.01$, therefore we have $Kz = 0.1$. We know that the closed loop poles $s = -1 \pm j$ lie on the root locus, hence

$$K = -\left.\frac{s(s+2)(s+0.01)}{(s+z)}\right|_{s=-1+j}$$

Solving for K and z, we get $K = 1.88$ and $z = 0.0532$. Therefore, the lag compensator is given by

$$D(s) = 1.88\frac{(s+0.532)}{(s+0.01)} \qquad ■$$