Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Chapter 6

Frequency Response Analysis

Frequency response methods are among the most useful techniques available for control system analysis and design. There is no one systematic design procedure for all control problems, rather, the different techniques complement each other. Root-locus techniques give powerful indicators for closed-loop transient response. Unfortunately, we need accurate, hence expensive, plant models to benefit from the root locus techniques. One of the advantages of the frequency response methods is that the response of the system can be obtained from measurements on the physical system without deriving the system transfer function. In fact, it is possible to design a control system without the need for a transfer function model.

In this chapter we consider frequency-response analysis methods, the important tools of Bode and Nyquist plots are presented.

6.1 Frequency response

In Chapter 3, the time responses of first and second order systems were considered. In this section we give meaning to steady-state response of systems to sinusoidal inputs, which is called the *frequency response*. Suppose that the input to a system with transfer function $G(s)$ is the sinusoid

$$r(t) = A\cos\omega t$$

Then

$$R(s) = \frac{As}{s^2+\omega^2}$$

and

$$Y(s) = G(s)R(s) = G(s)\frac{As}{(s-j\omega)(s+j\omega)}$$

We can expand this expression into partial fractions of the form

$$Y(s) = \frac{k_1}{s-j\omega} + \frac{k_2}{s+j\omega} + F(s) \tag{6.1}$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

where $F(s)$ is the collection of all terms in the partial fraction that originate in the denominator of $G(s)$. It is assumed the system poles are real, distinct and are in the LHP implying that the terms in $F(s)$ will decay to zero with increasing time. Therefore, only the first two terms in (6.1) contribute to the steady-state response. Using the cover-up method to find k_1 and k_2 we have

$$k_1 = (s - j\omega)Y(s)\big|_{s=j\omega} = \frac{1}{2}AG(j\omega)$$
$$k_2 = (s + j\omega)Y(s)\big|_{s=-j\omega} = \frac{1}{2}AG(-j\omega)$$

and k_2 is seen to be the complex conjugate of k_1. For any given value of ω, k_1 and k_2 are complex numbers and will find it convenient to express them in polar form as

$$k_1 = \frac{A}{2}|G(j\omega)|e^{j\phi}$$

where $|G(j\omega)|$ is the magnitude and $\phi = \angle G(j\omega)$. Then $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ and its sinusoidal steady-state value (i.e. $\lim_{t\to\infty} y(t)$) is

$$\begin{aligned} y_{ss}(t) &= k_1 e^{j\omega t} + k_2 e^{-j\omega t} \\ &= \frac{A}{2}|G(j\omega)|e^{j\phi}e^{j\omega t} + \frac{A}{2}|G(j\omega)|e^{-j\phi}e^{-j\omega t} \\ &= A|G(j\omega)|\frac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2} = A|G(j\omega)|\cos(\omega t + \phi) \end{aligned}$$

since $|G(j\omega)| = |G(-j\omega)|$.

This analysis can be summarized as follows. If a sinusoid input is applied to a system all of whose poles have a negative real part, the steady state response is a scaled, phase-shifted version of the input. The scaling factor called the *steady-state gain* is $|G(j\omega)|$ and the phase shift is the phase of $G(j\omega)$.

Example 6.1 Consider the system with the transfer function

$$G(s) = \frac{5}{s+2}$$

and an input $7\cos 3t$. Then

$$G(s)\big|_{s=j3} = \frac{5}{2+j3} = 1.387\angle{-56.3^\circ}$$

and the steady-state output is given by

$$y_{ss}(t) = (1.387)(7)\cos(3t - 56.3^\circ) = 9.79\cos(3t - 56.3^\circ) \qquad \blacksquare$$

We see then that, from the complex function $G(j\omega)$, we can obtain the steady-state response for any sinusoidal input, provided that the system is stable. We call $G(j\omega)$, $0 \le \omega \le \infty$, the *frequency response function.* We usually plot $G(j\omega)$ versus ω in some form to characterize the frequency response. We illustrate two forms by a simple example. Suppose we have a system with transfer function

$$G(s) = \frac{1}{s+1}$$

The frequency response function of this system is given

$$G(j\omega) = \frac{1}{1 + j\omega} = \frac{1}{\sqrt{1+\omega^2}} \angle -\tan^{-1}(\omega) \tag{6.2}$$

One common method of displaying this frequency response is in the form of a *polar plot.* In such a plot, the magnitude and angle of the frequency response function are plotted in the complex plane as the frequency, ω, is varied. For the function of (6.2), to construct a polar plot we first evaluate the function for values of ω. As an example, a table of these values is given in Table 6.1. Next

Table 6.1: Frequency response

ω	$G(j\omega)$
0	$1.000 \angle 0°$
0.5	$0.894 \angle -26.6°$
1.0	$0.707 \angle -45°$
1.5	$0.555 \angle -56.3°$
2.0	$0.447 \angle -63.4°$
3.0	$0.316 \angle -71.6°$
5.0	$0.196 \angle -78.7°$
10	$0.100 \angle -84.3°$
∞	$0.000 \angle -90°$

these values are plotted in the complex plane, as shown in Figure 6.1.

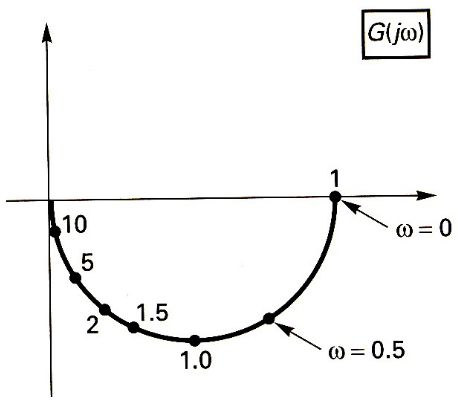

Figure 6.1: Frequency response.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Note that mathematically, the frequency response is a mapping from the s-plane to the $G(j\omega)$-plane. The upper half of the $j\omega$-axis which is a straight line, is mapped into the complex plane via the mapping $G(j\omega)$. A second form for displaying the frequency response is to plot the magnitude and phase of $G(j\omega)$ versus ω. These plots for the example above are shown in Figure 6.2.

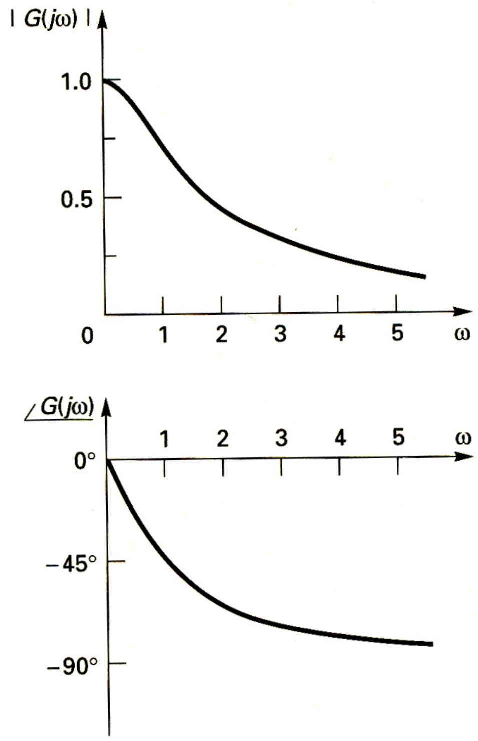

Figure 6.2: Frequency response.

6.2 Bode diagrams

This section presents a method for plotting a frequency response that is different from the two methods given in the first section of this chapter. This method results in a plot of magnitude versus frequency and phase versus frequency, but the frequency scale is logarithmic. In addition, the magnitude is also plotted on a logarithmic scale. The plot that is presented here is called a *Bode plot*, or a *Bode diagram.*

We develop the Bode diagram by using as an example the second-order transfer function

$$G(s) = \frac{K(1+\tau_3 s)}{(1+\tau_1 s)(1+\tau_2 s)} = \frac{K(1+s/\omega_3)}{(1+s/\omega_1)(1+s/\omega_2)} \tag{6.3}$$

where it is assumed that both poles and the zeros are real. Note that we have defined a constant ω_i equal to the reciprocal of τ_i. The reason for using the

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
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symbol for frequency will become evident later. Also, we call the value ω_i a *break frequency*, for a reason to be explained later.

First we form the magnitude of $G(j\omega)$

$$|G(j\omega)| = \frac{|K||1 + j\omega/\omega_3|}{|1 + j\omega/\omega_1||1 + j\omega/\omega_2|} \tag{6.4}$$

Next we use the property of logarithms given by

$$\log\left(\frac{ab}{cd}\right) = \log ab - \log cd = \log a + \log b - \log c - \log d$$

Also we define the unit decibel[1] (dB) as dB $= 20 \log a$, where a is a gain. A number-decibel conversion line is given in Figure 6.3. As a number increases by a factor of 10, the corresponding decibel value increases by a factor of 20. This

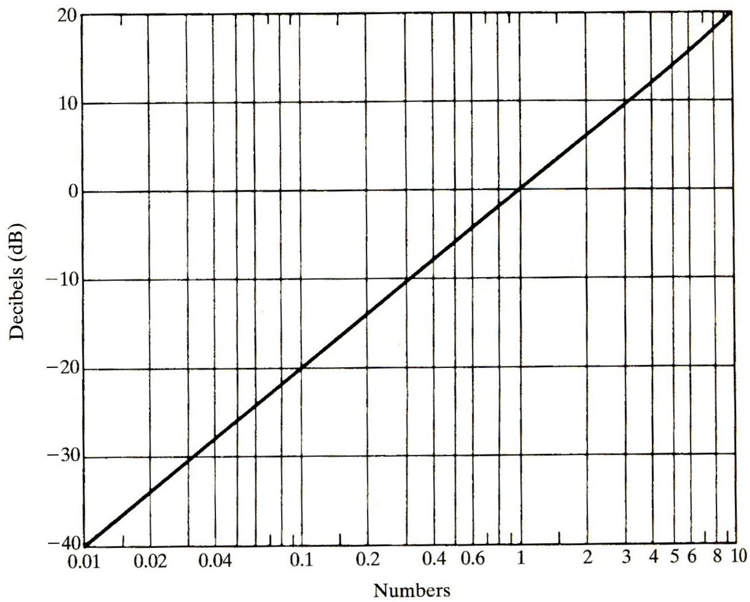

Figure 6.3: Number-decibel conversion.

may be seen from the following

$$20 \log(a \times 10^n) = 20 \log a + 20n$$

Note that when expressed in decibels, the reciprocal of a number differs from its value only in sign; that is,

$$20 \log a = -20 \log \frac{1}{a}$$

[1]The unit was first defined as **bel**, however, this unit proved to be too large, and hence a **decibel** (1/10 of a bel) was selected as a more useful unit.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

We plot the magnitude of the frequency response in decibels; that is we plot $20 \log |G(j\omega)|$. For the transfer function of (6.4),

$$20 \log |G(j\omega)| = 20 \log |K| + 20 \log \left|1 + \frac{j\omega}{\omega_3}\right| - 20 \log \left|1 + \frac{j\omega}{\omega_1}\right| - 20 \log \left|1 + \frac{j\omega}{\omega_2}\right| \quad (6.5)$$

The effect of plotting in decibels is then to cause the individual factors in the numerator to add to the total magnitude and the individual factors in the denominator to subtract from the total magnitude.

Consider now the general frequency dependent term in (6.5)

$$20 \log \left|1 + \frac{j\omega}{\omega_i}\right| = 20 \log \sqrt{\left[1 + \left(\frac{\omega}{\omega_i}\right)^2\right]} \quad (6.6)$$

This term is plotted versus $\log \omega$ in Figure 6.4. Note that the value of the term at the frequency ω_i (called the break frequency) is equal to $20 \log \sqrt{2} = 3.0103$. We

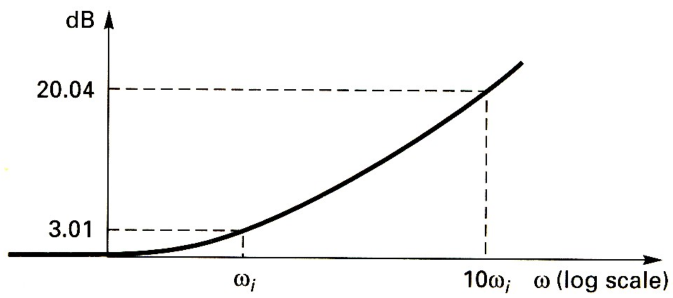

Figure 6.4: First-order term.

usually approximate this value as 3dB and say that, for a general first-oder numerator term, the value of the magnitude is equal to 3dB at the break frequency. For a first-order denominator term, the value is equal to −3dB at its break frequency. Note that the first-order term has a value of $20 \log \sqrt{101} = 20.04$, or approximately 20dB, at the frequency $10\omega_i$.

Accurate Bode diagrams are usually done using digital computers. However, there are situations in which approximate sketches of a Bode diagram are adequate. We now develop the approximations for the first-order terms. Consider the first order term of (6.6)

$$20 \log \sqrt{\left[1 + \left(\frac{\omega}{\omega_i}\right)^2\right]}$$

For frequencies very small compared to the break frequency ω_i, we have

$$20 \log(1) = 0 \qquad \omega \ll \omega_i$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

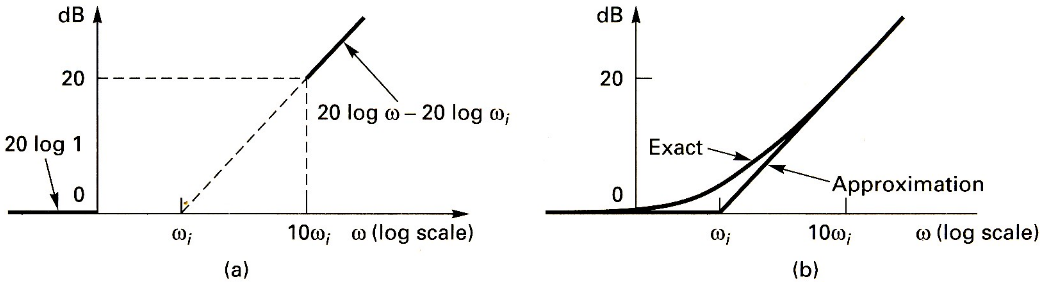

Figure 6.5: First-order approximation.

and for frequencies very large compared to ω_i,

$$20\log\left(\frac{\omega}{\omega_i}\right) = 20\log\omega - 20\log\omega_i \qquad \omega \gg \omega_i$$

For low frequencies the term is approximated by a straight line (the ω-axis). For high frequencies and if $\omega = 10\omega_i$ the difference between the logarithmic gains is 20dB. This represents a line that has a slope of 20dB per decade of frequency. Equating the above high-frequency and low-frequency expressions shows that the two straight lines intersect at $\omega = \omega_i$. The two terms are plotted in Figure 6.5(a). Comparing this figure with the exact curve of Figure 6.4, we see that the exact curve approaches the straight lines asymptotically, as is shown in Figure 6.5(b). As an approximation in sketching, we quite often extend the straight lines to the intersection at $\omega = \omega_i$ and use this straight line approximation instead of the exact curve. The frequency ω_i is called the *break frequency* because of the break in the slope at that frequency, as shown in Figure 6.5(b).

In constructing frequency responses, we consider the following types of transfer function factors:

1. Constant gain
2. Poles and zeros at the origin
3. Real poles and zeros not at the origin
4. Complex poles and zeros

We now consider each of these factors in order. First we develop the magnitude plots, and then we develop the phase plots.

6.2.1 Constant gain

For the case of a constant gain the magnitude is

$$20\log|K|$$

this term does not vary with frequency. The two possible cases are shown in Figure 6.6. If $|K|$ is greater than unity, the magnitude is positive; if $|K|$ is less than unity, the magnitude is negative. In either case, the magnitude plot is a straight line with a slope of zero.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

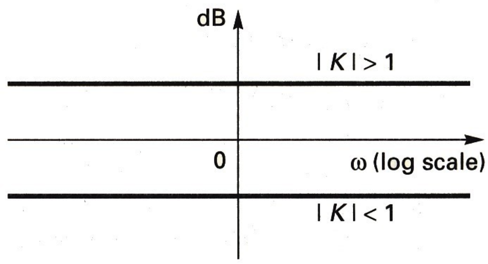

Figure 6.6: Constant-gain term.

6.2.2 Poles and zeros at the origin

For the case that the transfer function has a zero at the origin, the magnitude of this term is given by

$$20 \log |j\omega| = 20 \log \omega$$

Hence a plot of this term is a straight line, with a slope of 20dB per decade of frequency, that intersects the ω-axis at $\omega = 1$. The plot of this term is shown in Figure 6.7(a).

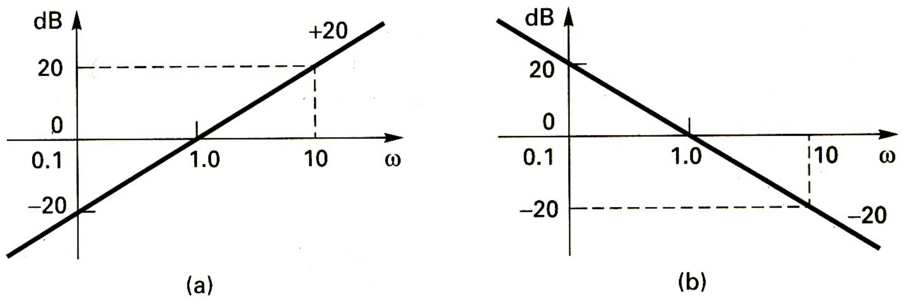

Figure 6.7: Zero and pole at $s = 0$.

For the case that the transfer function has a pole at the origin, the magnitude of the term is given by

$$20 \log \left| \frac{1}{j\omega} \right| = -20 \log \omega$$

and the curve is the negative of that for a zero at the origin. Thus the curve is a straight line with a slope of -20dB/decade that intersects the log ω axis at $\omega = 1$. This curve is shown in Figure 6.7(b). For the case of Nth-order zeros at the origin, the magnitude is

$$20 \log |(j\omega)^N| = 20 \log \omega^N = 20N \log \omega$$

Thus the curve is still a straight line that intersects the ω-axis at $\omega = 1$, but the slope is now $20N$ dB per decade. For example if we have two zeros at the origin the slope is 40dB/decade and -40dB/decade if two poles are at $s = 0$.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

6.2.3 Nonzero real poles and zeros

The case of real poles and zeros was considered previously. For a term of this type

$$20\log\left|1+\frac{j\omega}{\omega_i}\right| = 20\log\sqrt{\left[1+\left(\frac{\omega}{\omega_i}\right)^2\right]}$$
$$\approx \begin{cases} 0 & \omega \le \omega_i \\ 20\log\omega - 20\log\omega_i & w > \omega_i \end{cases}$$

This straight line approximation is shown in Figure 6.8(a) for a zero and in Figure 6.8(b) for a pole. Note that the terms have been normalized to have a dc gain of unity, or 0 dB. This is convenient otherwise each term will have a different low-frequency gain, and the Bode diagram is somewhat difficult to plot.

Suppose that first-order term is repeated, that is, suppose that we have an Nth-order term of the form $(1+s/\omega_i)^N$. The magnitude term is then given by

$$20\log\left[1+\left(\frac{\omega}{\omega_i}\right)^2\right]^{N/2} \approx \begin{cases} 0 & \omega \ll \omega_i \\ 20N\log\omega/\omega_i & \omega \gg \omega_i \end{cases} \tag{6.7}$$

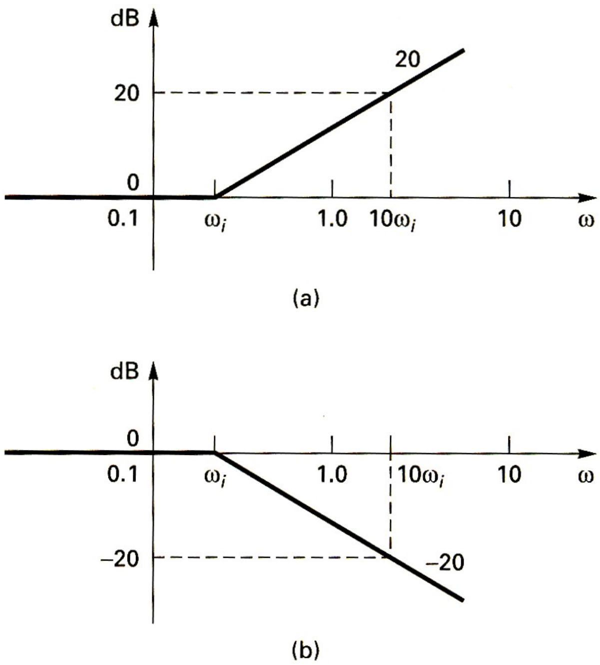

Figure 6.8: First-order terms.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
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The straight line approximation for this term is shown in Figure 6.9 for the case of a numerator term. It is seen that for $\omega > \omega_i$, the line has a slope of $20N$. For a given denominator term, the slope is $-20N$.

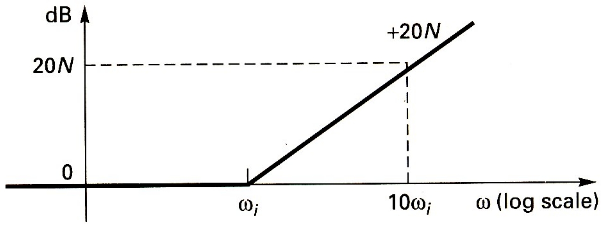

Figure 6.9: Bode diagram for repeated zeros.

Example 6.2

Plot the Bode diagram for the system with the transfer function

$$G(s) = \frac{10(s+1)}{(s+10)}$$

■ **Solution** First we convert the function to the form of (6.3)

$$G(s) = \frac{(1+s)}{(1+s/10)}$$

The break frequency of the numerator is $\omega = 1$, and the freak frequency of the denominator is $\omega = 10$. The numerator term, the denominator term, and the total magnitude (which, from (6.5), is the sum of the two terms) are shown in Figure 6.10. ■

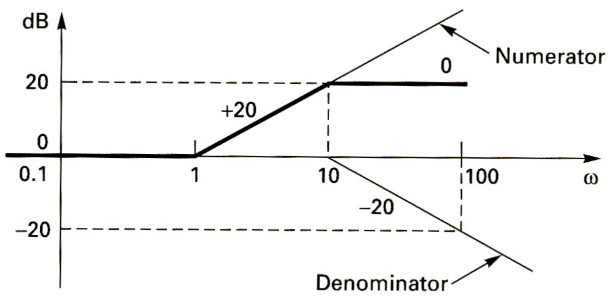

Figure 6.10: Bode diagram for Example 6.2.

Example 6.3

As a second example, consider the transfer function

$$G(s) = \frac{200(s+1)}{(s+10)^2}$$

■ **Solution** We rewrite the transfer function as

$$G(s) = \frac{2(1+s)}{(1+s/10)^2}$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

The bode diagram has three terms. The first term is the constant gain, which adds a term of value $20 \log 2 = 6\text{dB}$ at all frequencies. The second term is the zero term with break frequency at $\omega = 1$, and the third term is the second-order pole at $\omega = 10$. The three terms are plotted in Figure 6.11. ■

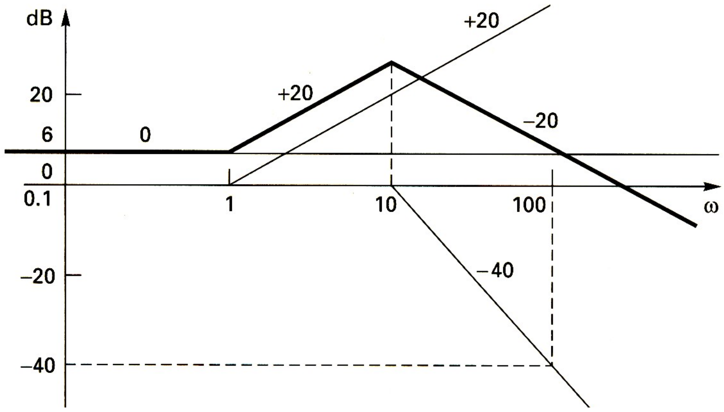

Figure 6.11: Bode diagram for Example 6.3.

Example 6.4

A third example, with

$$G(s) = \frac{1000(s+3)}{s(s+12)(s+50)}$$

■ **Solution** The transfer function is rewritten as

$$G(s) = \frac{5(1+s/3)}{s(1+s/12)(1+s/50)}$$

The constant gain term is obtained from

$$\frac{(1000)(3)}{(12)(50)} = 5$$

The constant gain term is now $20 \log 5 = 14\text{dB}$, and the five terms of the Bode diagram are as shown in Figure 6.12.

6.2.4 Phase diagrams

Before we consider the final type of terms that can appear in a Bode diagram, we construct the phase diagrams for the three types of terms already considered. First, for the constant gain term, the phase angle is either $0°$ or $\pm 180°$. If the gain term is positive, the phase angle is $0°$; if the gain term is negative, the phase angle can be plotted as either $180°$ or $-180°$. For a zero of the transfer function at the origin, the phase angle is $90°$, since

$$s|_{s=j\omega} = j\omega = \omega \underline{/90°}$$

In a like manner, a pole at the origin gives a phase angle of $-90°$, since

$$\left.\frac{1}{s}\right|_{s=j\omega} = \frac{1}{j\omega} = \frac{1}{\omega} \underline{/-90°}$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

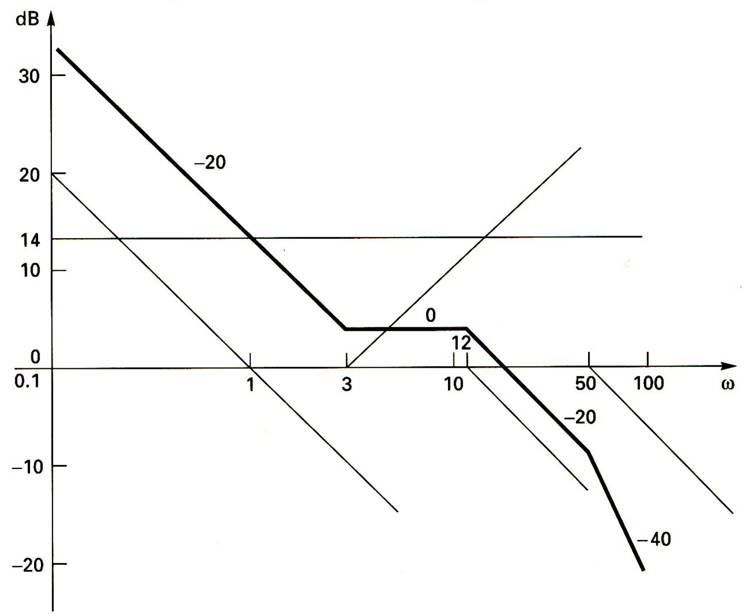

Figure 6.12: Bode diagram for Example 6.4.

For a real zero of the transfer function, with the zero not at the origin, the term is given by

$$\left(1+\frac{s}{\omega_i}\right)\bigg|_{s=j\omega} = 1+\frac{j\omega}{\omega_i} = \sqrt{1+\left(\frac{\omega}{\omega_i}\right)^2}\ \underline{/\Theta(\omega)}$$

where

$$\Theta(\omega) = \tan^{-1}\left(\frac{\omega}{\omega_i}\right)$$

Figure 6.13 shows the phase Θ plotted for various values of the ratio ω/ω_i. The exact curve is approximatted with the straight line shown in Figure 6.13. The straight line approximation for the phase characteristic breaks from $0°$ at the frequency $0.1\omega_i$ and breaks back to the constant value of $90°$ at $10\omega_i$. Note that the phase characteristic for a pole is the negative of that for a zero, since, for a pole,

$$\frac{1}{1+s/\omega_i}\bigg|_{s=j\omega} = \frac{1}{1+j\omega/\omega_i} = \frac{1}{\sqrt{1+(\omega/\omega_i)^2}}\ \underline{/\Theta(\omega)}$$

where

$$\Theta(\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_i}\right)$$

Example 6.5 Consider again the system of Example 6.1

$$G(s) = \frac{(1+s)}{(1+s/10)}$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

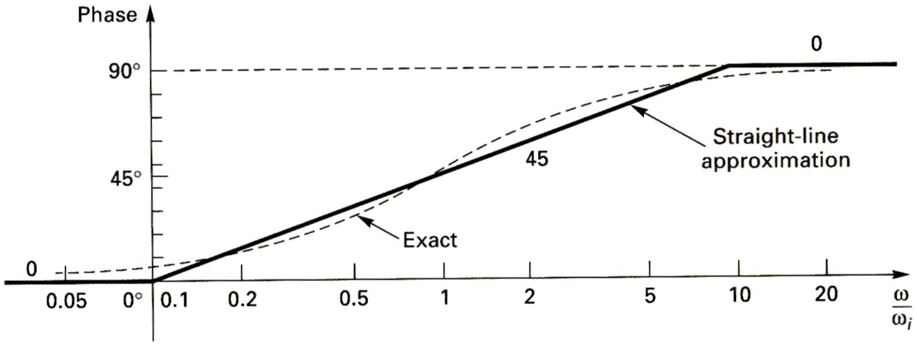

Figure 6.13: Phase characteristics of a real zero.

■ **Solution** For the zero $\omega_i = 1$, hence the straight line approximation to the phase characteristic breaks at $\omega = 0.1(1) = 0.1$ and breaks back at $\omega = 10(1) = 10$. The pole has $\omega_i = 10$ and breaks at $\omega = 0.1(10) = 1$ and breaks at $\omega = 10(10) = 100$. These characteristics, along with the total phase diagram (which is the sum of the two characteristics), are plotted in Figure 6.14.

Example 6.6

As a second example illustrating the phase characteristic of the Bode diagram, consider the system of Example 6.3, with the transfer function

$$G(s) = \frac{5(1 + s/3)}{s(1 + s/12)(1 + s/50)}$$

The phase characteristics of the various terms, along with the total phase characteristic of the system, are given in Figure 6.15. ■

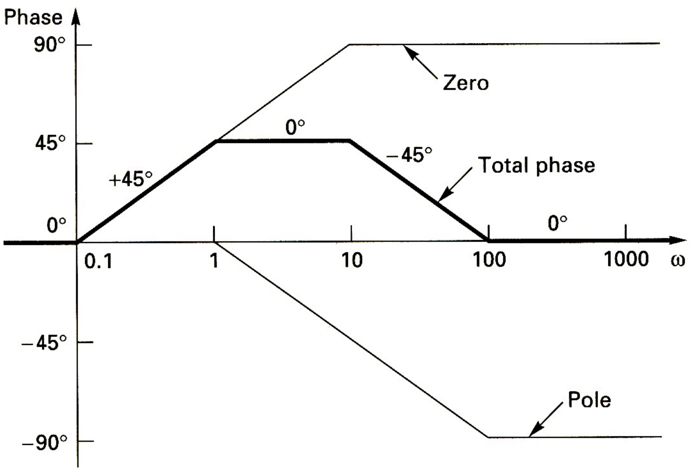

Figure 6.14: Example 6.5.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
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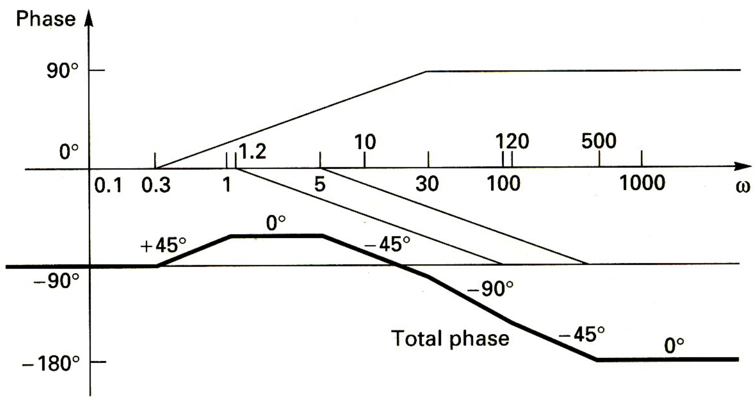


Figure 6.15: Bode phase diagram for Example 6.6.

Example 6.7 As a final example, the complete Bode diagram will be constructed for the transfer function

$$G(s) = \frac{(1-s)}{(1+s/10)}$$

For the zero at $s = 1$,

$$1 - j\omega = \sqrt{(1+\omega^2)}\ \underline{/\Theta(\omega)} \qquad \Theta(\omega) = \tan^{-1}(-\omega)$$

The magnitude characteristic is as shown in Figure 6.16. The phase of the zero term varies from 0° to −90°, because of the minus sign on the imaginary part. The total phase characteristic is then as shown in Figure 6.16. The characteristics at the extremes in frequency are verified through the calculations

$$\lim_{\omega\to 0} G(j\omega) = 1\ \underline{/0^\circ}$$

$$\lim_{\omega\to\infty} G(j\omega) = \frac{-j\omega}{j\omega/10} = -10 = 10\ \underline{/-180^\circ} \qquad \blacksquare$$

6.2.5 Complex poles and zeros

In this section we consider an additional term that can be encountered in constructing a Bode diagram. We consider poles or zeros of the form

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \qquad 0 \le \zeta < 1 \tag{6.8}$$

For convenience in plotting we normalize (6.8), to have a dc gain of unity; this is accomplished by factoring out ω_n^2. Hence, we consider

$$1 + 2\zeta\frac{s}{\omega_n} + \left(\frac{s}{\omega_n}\right)^2 \tag{6.9}$$

The magnitude and phase of this expression for $s = j\omega$ is an involved function of the damping ratio ζ, and in general it does not lend itself to approximations

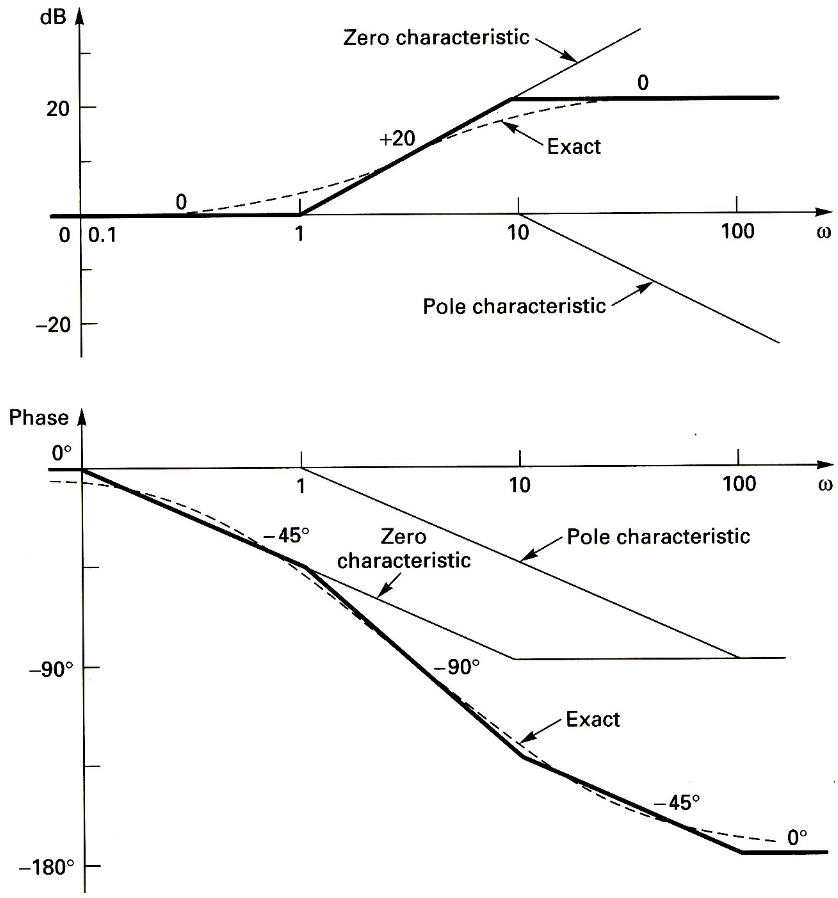


Figure 6.16: Bode diagram of Example 6.7.

by straight lines.

Consider first the case that $\zeta = 1$. For this case, (6.9) has two real equal zeros

$$1 + 2\zeta\frac{s}{\omega_n} + \left(\frac{s}{\omega_n}\right)^2\bigg|_{\zeta=1} = \left(1 + \frac{s}{\omega_n}\right)^2 \tag{6.10}$$

Since the zeros are real, this case is covered by the methods given in the preceding sections. The straight-line approximations for this case are given in Figure 6.17, along with the exact curves. For cases in which $0 < \zeta < 1$, the asymptotic approximations to the frequency response curves are not accurate and the errors can be large for low values of ζ. This is because the magnitude and phase of (6.9) depend on both the break frequency and the damping ratio ζ. Noting that the exact magnitude of (6.9) in dB is

$$20\log\left|1 + 2\zeta\frac{j\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right| = 20\log\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

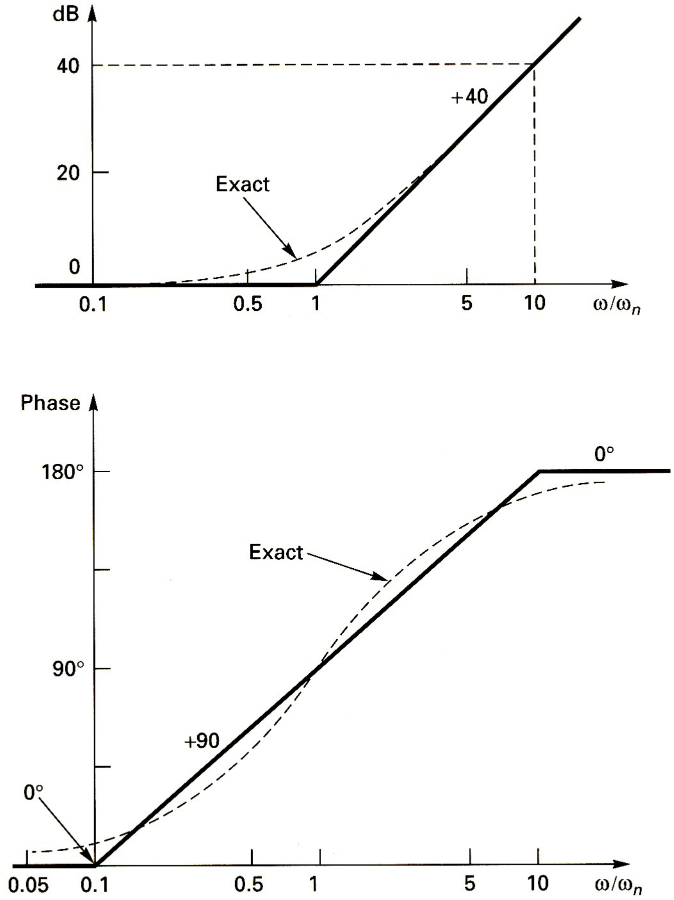


Figure 6.17: Bode diagrams of a repeated zero.

The approximation to the frequency response may be obtained as follows: for low frequencies such that $\omega \ll \omega_n$, the magnitude becomes

$$20 \log 1 = 0 \text{ dB}$$

The low-frequency asymptote is thus a horizontal line at 0 dB. For $\omega \gg \omega_n$, the magnitude becomes

$$20 \log \frac{\omega^2}{\omega_n^2} = 40 \log \frac{\omega}{\omega_n}$$

the equation for the high frequency asymptote is a straight line having the slope -40 dB/decade. The high-frequency asymptote intersects the low-frequency one at $\omega = \omega_n$. The two asymptotes just derived are independent of the value of ζ. The phase for second order zeros is

$$\tan^{-1}\left(\frac{2\zeta \dfrac{\omega}{\omega_n}}{1 - \dfrac{\omega^2}{\omega_n^2}}\right) \approx \begin{cases} 0 & \omega \ll \omega_n \\ 180^\circ & \omega \gg \omega_n \end{cases}$$

Figure 6.18 illustrates some exact curves for several values of ζ between zero and unity for complex zeros. Once again, the curves for complex poles are obtained by inverting these curves.

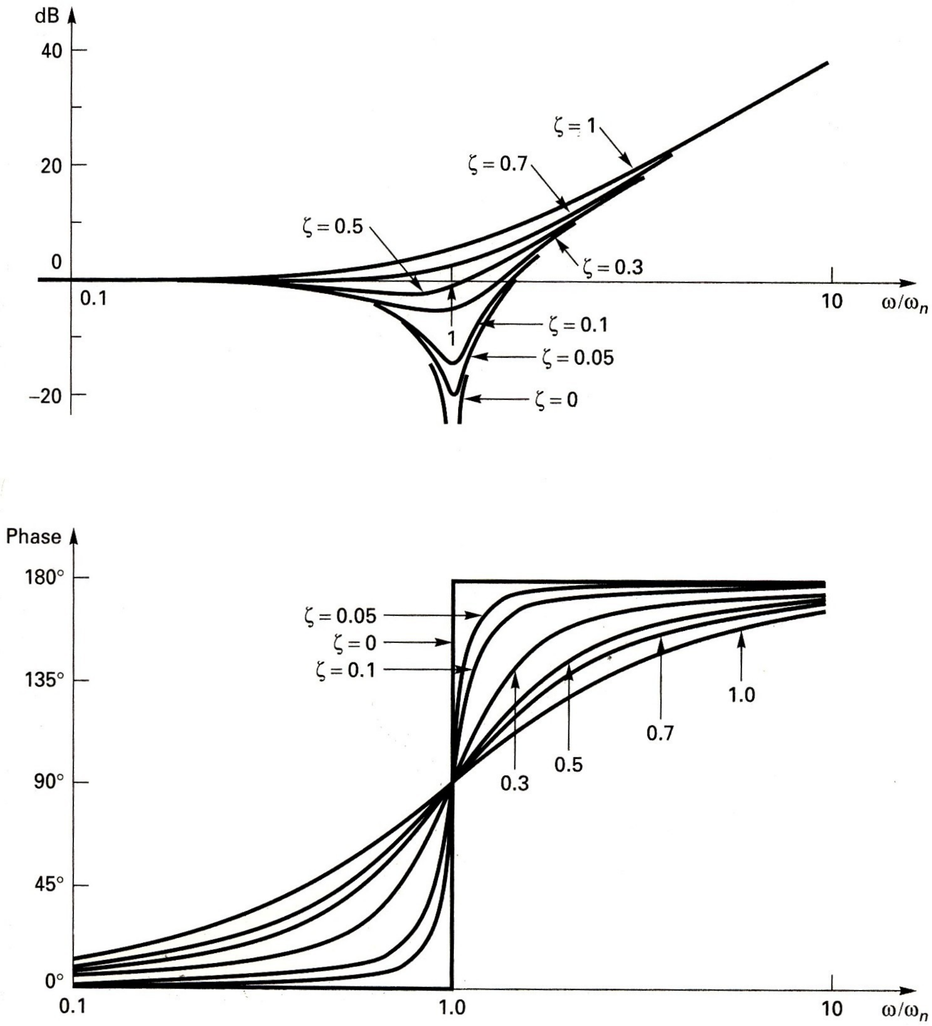


Figure 6.18: Bode diagrams of complex zeros.

For the case that $\zeta < 0.3$, the straight line approximations are very inaccurate and are seldom used. Instead exact curves such as in Figure 6.18 are used. An example is now given to illustrate complex terms ina Bode diagram.

Example 6.8

Consider the transfer function

$$G(s) = \frac{200(s+1)}{s^2 + 4s + 100} = \frac{2(s+1)}{(s/10)^2 + 2(0.2)(s/10) + 1}$$

For the complex poles, $\zeta = 0.2$ and $\omega_n = 10$. We do not expect the straight-line approximation to be very accurate. Both the straight line approximation and

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

the exact Bode diagram are given in Figure 6.19. The maximum error in the magnitude diagram for the straight line approximation is seen to be approximately 8 dB. Note also the very large errors in the straight line approximation for the phase.

6.3 Nyquist plots

Frequency response information can be presented in a various forms of which Bode plot is one. The same information can be presented by the *Nyquist* plot also known as the polar plot. The Nyquist plot of the frequency response function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity. An example of such a plot is shown in Figure 6.20. The projection of $G(j\omega)$ on the real and imaginary axes are its real and imaginary components. An advantage in using a Nyquist plot is that it depicts the frequency response characteristics of a system over

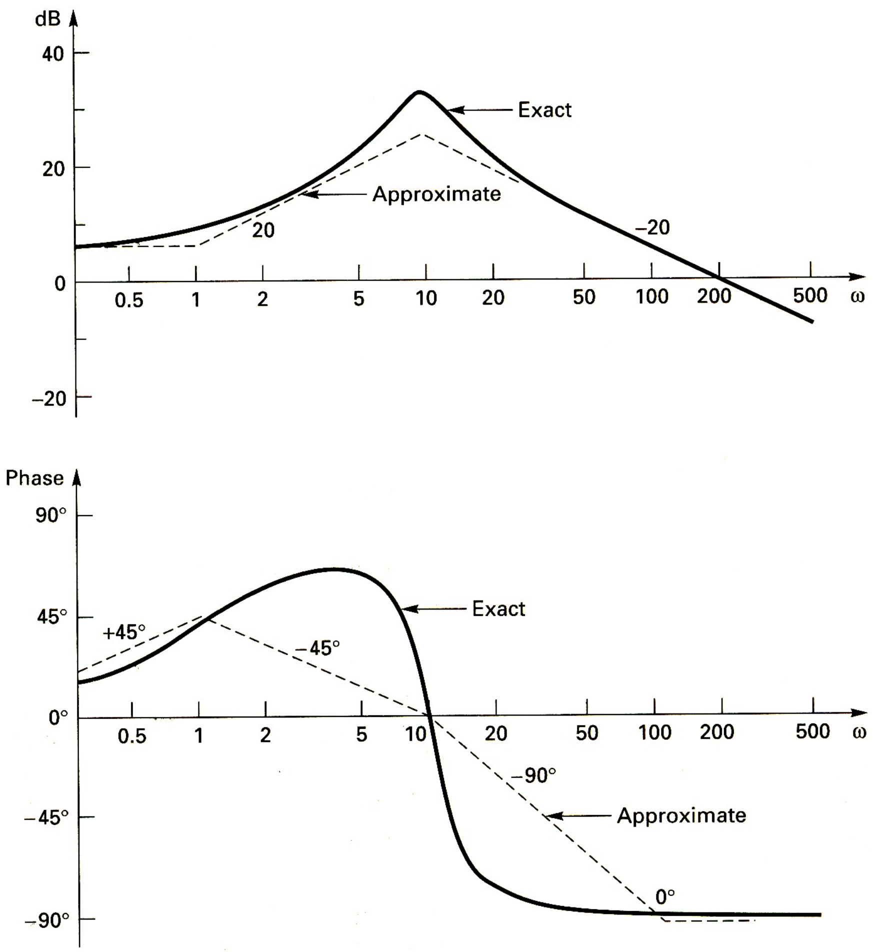


Figure 6.19: Bode diagram for Example 6.8.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

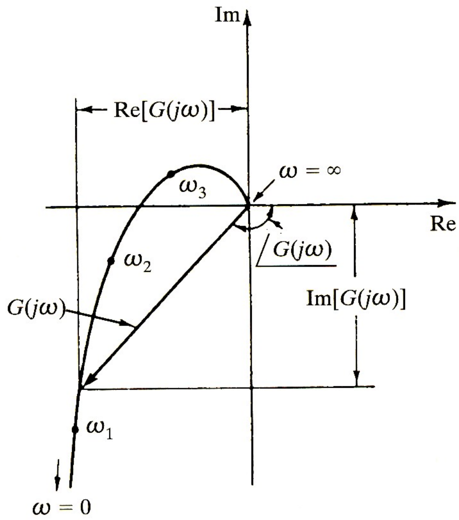


Figure 6.20: Nyquist plot.

the entire frequency range in a single plot. Table 6.2 shows examples of Nyquist plots of simple transfer functions.

6.3.1 Nyquist criterion

In designing a control system, we require that the system is stable. In what follows we shall show that the Nyquist plot indicates not only whether a system is stable but also the degree of stability of a stable system. In this section we consider closed loop systems of the type shown in Figure 6.21. The closed loop

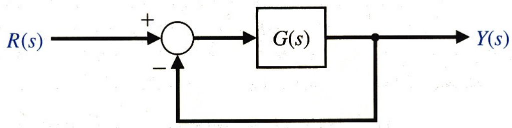


Figure 6.21: Closed loop system.

transfer function is given by

$$H(s) = \frac{G(s)}{1 + G(s)}$$

and the characteristic equation is given by

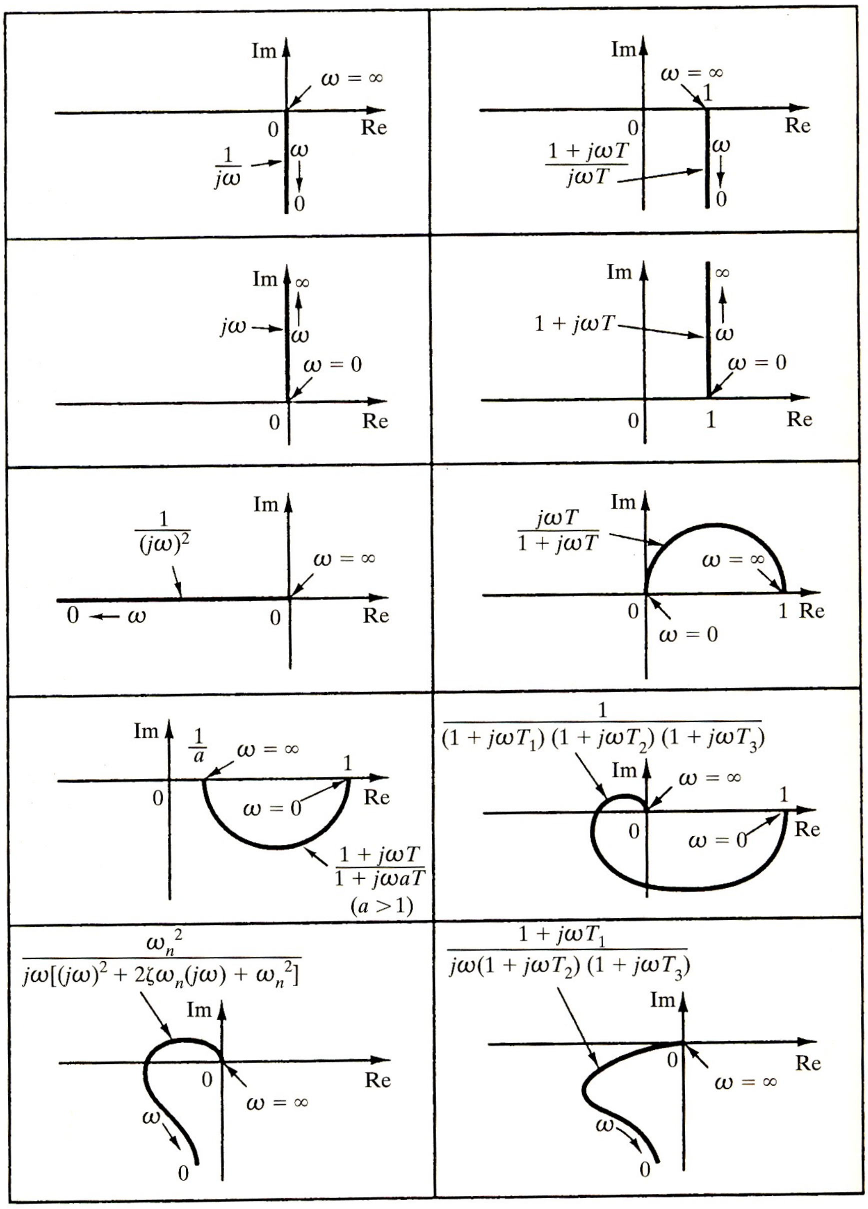
$$1 + G(s) = 0$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

We can write the closed loop transfer function as

$$H(s) = \frac{G(s)}{1+G(s)} = \frac{n(s)/d(s)}{1+n(s)/d(s)}$$
$$= \frac{n(s)}{d(s)+n(s)}$$

Table 6.2: Nyquist plots of simple transfer functions.



Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
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Furthermore, we can write the characteristic equation as

$$F(s) = 1 + G(s) = 1 + \frac{n(s)}{d(s)} = 0$$

$$\Longrightarrow \; F(s) = \frac{d(s) + n(s)}{d(s)} = 0$$

That is

1 **The poles of $F(s)$ are the open loop poles (i.e., poles of $G(s)$).**

2 **The zeros of $F(s)$ are the closed loop poles (i.e., poles of $H(s)$).**

Therefore, to determine closed loop stability, we need to know the number of right-half plane zeros of $F(s)$.

In order to introduce the Nyquist criterion, we consider some mappings from the complex s-plane to the $F(s)$-plane. For example consider the function

$$F(s) = \frac{s - 0.5}{s(s-1)(s+4)}$$

Suppose F is evaluated around the simple, circular closed contour Ω of radius 2 in the s-plane as shown in Figure 6.22(a). Evaluating F at each point on Ω generates the closed loop contour Γ shown in Figure 6.22(b). Table 6.3 provides the values of F at some key points along the contour Ω. The closed contour Γ could then be approximated by simply plotting and connecting these points. It is woth noting thet Γ as shown in Figure 6.22 is not drawn to scale.

Note that the contour in the s-plane, where F was evaluated, was traversed in the counterclockwise direction, and enclosed the circular region in the s-plane. Further, the contour Γ generated by the evaluation of F along Ω evolves in the clockwise direction. Also note the contour Γ encircles the origin of the F-plane.

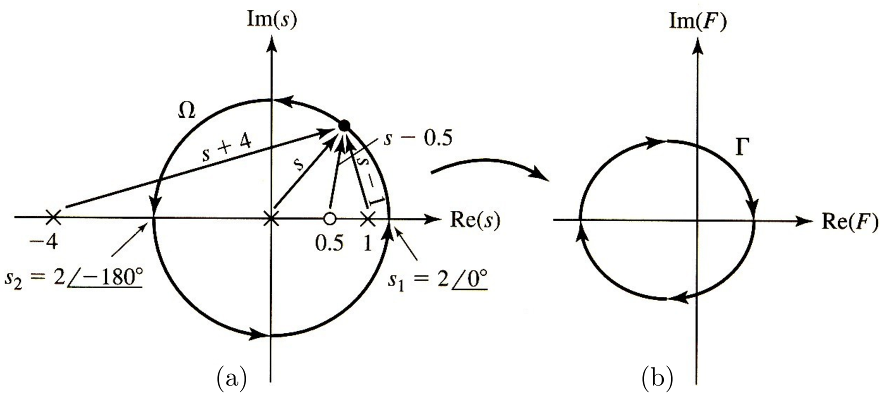

Figure 6.22: (a) Curve Ω in the s-plane and (b) resulting curve Γ in the F-plane.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Table 6.3: Magnitude and phase of $F(s)$ along Γ.

θ	$F(2\underline{/\theta})$
$0°$	$0.125\ \underline{/0°}$
$20°$	$0.13\ \underline{/-20°}$
$40°$	$0.16\ \underline{/-48°}$
$60°$	$0.17\ \underline{/-81°}$
$80°$	$0.18\ \underline{/-180°}$
$100°$	$0.2\ \underline{/-138°}$
$120°$	$0.19\ \underline{/-162°}$
$140°$	$0.19\ \underline{/-175°}$
$160°$	$0.2\ \underline{/-179.7°}$
$180°$	$0.21\ \underline{/180°}$

Based on the above observations, one can ask is there a relationship between the number of poles and zeros encircled by Ω in the s-plane and the number and direction of encirclements of the origin in the F-plane. In the above mapping, the counterclockwise encirclement of two poles and one zero resulted in one clockwise encirclement of the origin.

The relationship between the contours in the two complex planes is given by *Cauchy's theorem* (known as Cauchy's principle of argument) twhich states (given here without proof) "*for a given contour in the s-plane that encircles P poles and Z zeros of the function $F(s)$ in a clockwise direction, the resulting contour in the F-plane encircles the origin a total of N times in a clockwise direction, where $N = Z - P$*". This theorem explains the mapping in Figure 6.22, since $Z = 1$ and $P = 2$, hence, $N = -1$. Therefore, the contour Γ encircles the origin once and the negative sign implies opposite direction to the contour Ω.

We now develop the Nyquist criterion. Suppose that we let the mapping of $F(s)$ be the characteristic polynomial of the closed-loop system of Figure 6.21; that is;

$$F(s) = 1 + G(s)$$

Furthermore, let the curve Ω be as shown in Figure 6.23(a). This curve, which is composed of the imaginary axis and an arc of in finite radius, completely encircles the right half of the s-plane. Then, in Cauchy's principle of argument, Z is the number of zeros of the system characteristic polynomial in the right half of the s-plane. Also recall that Z is the number of poles of the closed-loop

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

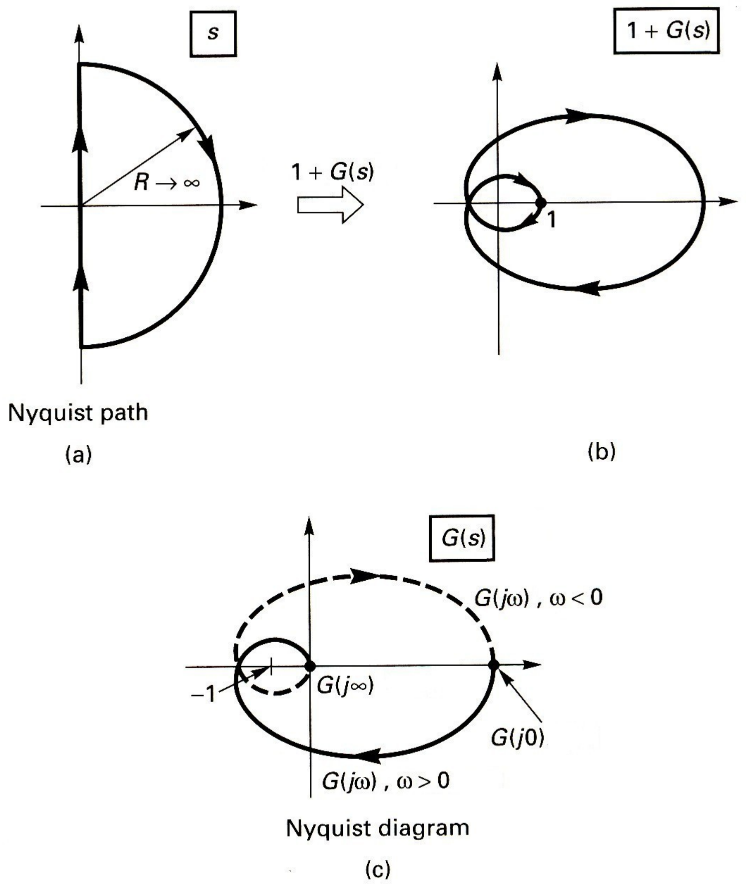

Figure 6.23: Nyquist diagram.

system in the RHP. *Therefore, Z must be zero for the system to be stable.* P is the number of poles of the characteristic polynomial in the right half of the s-plane and thus is the number of poles of the open loop function $G(s)$ in the right half plane, since the poles of $1 + G(s)$ are also those of $G(s)$.

The curve in Figure 6.23(a) is called the *Nyquist path*, and a typical mapping is shown in Figure 6.23(b). The mapping encircles the origin two times in the clockwise direction, and from Cauchy's principle

$$N = 2 = Z - P$$

or

$$Z = 2 + P$$

Since P is the number of poles of a function inside the Nyquist path, it cannot be a negative number. This in this example, Z is greater than or equal to 2, and the closed loop system is unstable.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

To simplify the application of the Nyquist criterion a modification is usually made. Instead of plotting $1 + G(s)$, as in Figure 6.23(b), we plot just $G(s)$. The resulting plot has the same shape but is shifted one unit to the left, as shown in Figure 6.23(c). Hence, rather than plotting $1 + G(s)$ and counting encirclements of the origin, we get the same result by plotting $G(s)$ and counting encirclements of the point $-1 + j0$. The resultant plot of the open-loop function $G(s)$ is called the *Nyquist diagram.* Note that we are plotting the open-loop function to determine closed-loop stability.

A simple example is given to illustrate the Nyquist criterion.

Example 6.9 Consider the system with open-loop transfer function

$$G(s) = \frac{5}{(s+1)^3}$$

Then

$$G(j\omega) = \frac{5}{(1+j\omega)^3}$$

An evaluation of this function is given in Table 6.4 for certain values of ω, and a plot of theses values is shown in Figure 6.24. The dc gain, $G(0)$, is equal to 5 and is shown as part I. The solid curve, part II, is obtained directly by plotting the values of Table 6.4. However, note as ω is increased the magnitude of each factor in the denominator has an increasing magnitude. Therefore $|G(j\omega)|$ decreases from 5 to 0. On the other hand, the angle of each factor in the denominator increases with ω from $0°$ to $90°$. Therefore, $\underline{/G(j\omega}$ decreases from $0°$ to $-270°$. To evaluate N(number of clockwise encirclements of -1) we need to calculate the intersection with the real axis, the point $G(j\omega_1)$. This can be calculated using the Routh-Hurwitz stability ciretrion as follows:

Table 6.4: Frequency response

ω	$G(j\omega)$
0	$5.00 \underline{/0°}$
0.5	$3.58 \underline{/-79.8°}$
1.0	$1.77 \underline{/-135°}$
1.5	$0.85 \underline{/-169°}$
2.0	$0.45 \underline{/-190.3°}$
5.0	$0.038 \underline{/-236.1°}$
20	$0.0006 \underline{/-261.3°}$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
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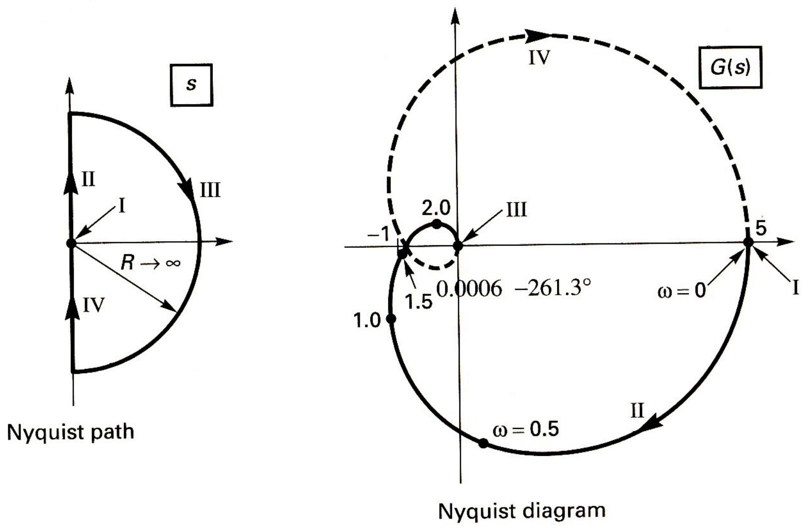

Figure 6.24: Nyquist diagram for Example 6.9.

1. Introduce a gain K into $G(s)$ and evaluate the characteristic equation

$$1 + KG(s) = 1 + \frac{5K}{(s+1)^3} = 0$$
$$\Longrightarrow s^3 + 3s^2 + 3s + 1 + 5K = 0$$

2. Form the Routh array

$$\begin{array}{c|ccl} s^3 & 1 & 3 & \\ s^2 & 3 & 1+5K & \\ s^1 & (8-5K)/3 & 0 & \Longrightarrow K < 1.6 \\ s^0 & 1+5K & 0 & \Longrightarrow K > -0.2 \end{array}$$

It is important to note here that the Nyquist diagram in Figure 6.24 is for $K = 1$. Thus the system is stable. However, we are interested in the frequency ω_1.

3. When $K = 1.6$, the system is marginally stable (the Nyquist diagram intersects the -1 point). Thus $1.6 \times G(j\omega_1) = -1$ and

$$G(j\omega_1) = -1/1.6 = -0.625$$

The frequency response intersects the negative real axis at the point -0.625, therefore, the -1 point is not encircled by the frequency response. Hence, $N =$ (number of clockwise encirclements of -1) $= 0$.

4. To find ω_1, we form the auxiliary equation

$$3s^2 + 1 + 5 \times 1.6 = 3s^2 + 9 = 3(s^2 + 3)$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

The roots are $s = \pm j\sqrt{3} = \pm j\omega_1$. We can check the value of the frequency response at ω_1 as:

$$G(j\sqrt{3}) = 5/(1 + j\sqrt{3})^3 = -0.625$$

This value checks that derived earlier.

Finally to apply the Nyquist criterion for

$$G(s) = \frac{5}{(s+1)^3}$$

The Nyquist equation is

$$Z = N + P$$

The value of P = (number of unstable poles of $G(s)$) = 0, since $G(s)$ has no poles in the right halh plane. The number of encirclements of the -1 point in the $G(s)$-plane, N, is zero. Then the number of unstable closed loop poles is given by

$$Z = N + P = 0$$

and the closed loop system is stable.

The Nyquist criterion is very important, since it allows us to determine the stability of the closed loop system from the knowledge of the frequency response of the open loop function.

6.3.2 Poles at the origin

In the preceding section, the condition for the application of Cauchy's principle of argument to determine stability is that the open loop function, which we assume to be $G(s)$, has no poles or zeros on the Nyquist path. When $G(s)$ has a pole at the origin of the s-plane (at $s = 0$), this point will be on the Nyquist path. It is still possible to determine closed loop stability, but the Nyquist contour must be modified. We simply introduce a detour in the Nyquist path

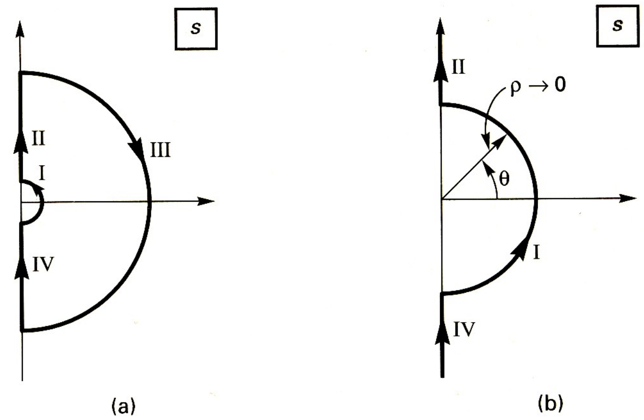

Figure 6.25: Detour around poles at the origin.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
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around the origin if $G(s)$ has a pole there. The resulting path is shown in Figure 6.25(a). This detour, shown as part I in the figure, is chosen to be very small, so that no right half-plane poles or zeros can occur within the region excluded by the detour. This is illustrated in Figure 6.25 (b), where the vicinity of the origin is shown with a greatly expanded scale. The detour is chosen to be circular with a radius that approaches zero in the limit. Then, on the detour,

$$s = \lim_{\rho \to 0} \rho e^{j\theta}, \quad -90^\circ \leq \theta \leq 90^\circ$$

Since $G(s)$ has poles at the origin, the magnitude of $G(s)$ will be very large on the detour of the Nyquist path. However, we will still be able to sketch the Nyquist diagram, but not to scale.

Example 6.10

Suppose $G(s)$ has the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

We begin the Nyquist path at $s = \rho$, that is, on the detour with $\theta = 0^\circ$, and let θ increases. On part I of the Nyquist path,

$$G(s)|_{s=\rho e^{j\theta}} = \frac{1}{\rho e^{j\theta}(\rho e^{j\theta}+1)} \quad 0^\circ \leq \theta \leq 90^\circ$$

and

$$\lim_{\rho \to 0} G(\rho e^{j\theta}) = \lim_{\rho \to 0} \frac{1}{\rho e^{j\theta}} = \lim_{\rho \to 0} \frac{1}{\rho} \angle{-\theta}$$

Hence part I of the nyquist path generates a very large arc on the Nyquist diagram, as shown in Figure 6.26. This arc is swings past the -90°-axis slightly, since the pole at $s = -1$ contributes a very small negative angle to the Nyquist diagram. Recall that the Nyquist diagram cannot be drawn to scale.

For part II of the Nyquist path, s is equal to $j\omega$, and this portion of the Nyquist diagram is a plot of the function

$$G(j\omega) = \frac{1}{j\omega(j\omega+1)}$$

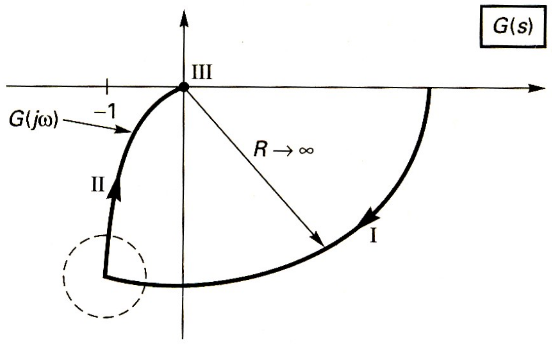

Figure 6.26: Nyquist diagram.

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The University of Tobruk
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The magnitude function $|G(j\omega)$ decreases from a very large value to zero, and the angle function $\underline{/G(j\omega)}$ decreases from a value slightly more negative than $-90°$ to $-180°$. The resulting Nyquist plot is shown as part II of Figure 6.26.

Note again that the Nyquist diagram cannot be drawn to scale for this example, since we must show very large magnitudes and on the same figure show the -1 point. The shape of the Nyquist diagram in the region shown enclosed by a dashed circle is not important, since this region is at a very large magnitude; the shape of the Nyquist diagram in the region cannot affect the number of encirclements of the -1 point. Finally applying the Nyquist criterion for this example

$$Z = N + P = 0 + 0 = 0$$

and the system is stable for all gains K, $K > 0$.

Example 6.11

As a second example of an open-loop function with poles at the origin, consider the transfer function

$$G(s) = \frac{1}{s^2(s+1)}$$

Since the transfer function has two poles at the origin, the Nyquist path must detour around the origin as shown in Figure 6.27(a). On this detour (part I),

$$G(s)|_{s=\rho e^{j\theta}} = \frac{1}{\rho^2 e^{j2\theta}(\rho e^{j\theta}+1)} \qquad 0° \le \theta \le 90°$$

and

$$\lim_{\rho\to 0} G(\rho e^{j\theta}) = \lim_{\rho\to 0} \frac{1}{\rho^2 e^{2j\theta}} = \lim_{\rho\to 0} \frac{1}{\rho^2} \underline{/-2\theta}$$

Thus the magnitude of the function is very large, and its angle rotates from $0°$ to slightly past $-180°$, as shown in Figure 6.27(b). This rotation past $-180°$-axis occurs because the pole at $s = -1$ contributes very small negative angle to the function. For part II of the Nyquist path, as ω increases from a very small value to a very large value, the magnitude function decreases from a very large value

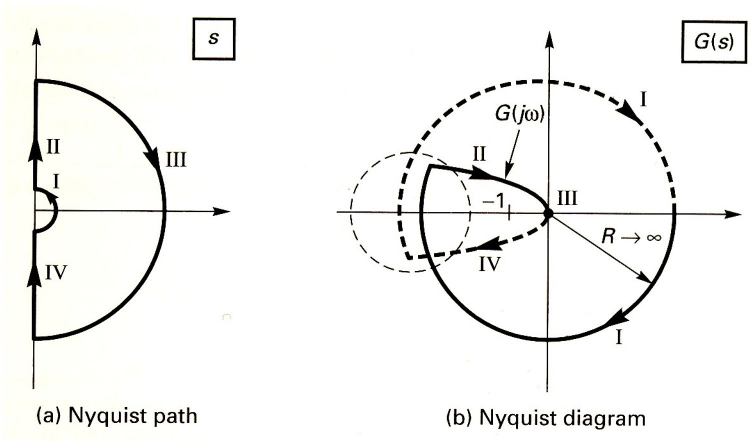

Figure 6.27: Nyquist diagram for Example 6.11.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
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to zero, and the angle decreases from $-180°$ to $-270°$. The resulting Nyquist diagram is then as shown in Figure 6.27(b). As shown in the previous example, the shape of the Nyquist diagram in the region enclosed by the dashed circle is not important, since the shape does not affect the number of encirclements. From the diagram we see that there are two encirclements of the -1 point and hence $N = 2$. Since the open-loop transfer function has no poles inside the Nyquist path, $P = 0$ and

$$Z = N + P = 2 + 0 = 2$$

Thus the closed loop transfer function has two unstable poles.

Difficulties can occur in counting the number of encirclements of the -1 point for complex Nyquist diagrams. For example, how many encirclements of the -1 point occur for the Nyquist diagram in Figure 6.28? The easiest way is to draw a ray from the -1 point in any convenient direction. One such ray is shown in Figure 6.27. The number of clockwise encirclements of the -1 point

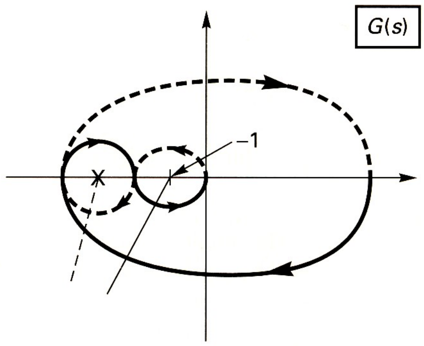

Figure 6.28: Counting encirclements on the Nyquist diagram.

is then is then equal to the number of crossings of this ray by the Nyquist diagram, in the clockwise direction, minus the number of crossings of the ray in the counterclockwise direction. For Figure 6.28, there is one crossing in each direction, and thus the number of encirclements is zero. If the -1 point were at a point x in this figure rather than at the point shown, then there are two clockwise crossings and no counterclockwise crossings. For this case, the Nyquist diagram has two clockwise encirclements of the -1 point.

6.4 Relative Stability

As seen in the previous section, the Nyquist diagram was used to determine if a system was stable or unstable. Although stability by itself is an important issue, an acceptable transient response of a system is not less important. Generally, we require not only that a system be stable but also that it be stable by some degree of safety.

We define the *relative stability* of a system in terms of the closeness of the

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Nyquist diagram to the −1 point in the complex plane. Two such measures are commonly used, the *gain* and *phase* margins.

6.4.1 The gain Margin

The gain margin is the factor by which the open-loop gain of a stable system must be changed to make the system marginally stable. In Figure 6.29, we denote the value of the Nyquist diagram at the $-180°$ crossover as $-\alpha$. If the open-loop function is multiplied by the gain of $K = 1/\alpha$, the Nyquist diagram intersects the −1 point, and the closed-loop system is marginally stable. The $1/\alpha$ value is called the *gain margin* and is usually given in decibels. If the Nyquist diagram has multiple $-180°$ crossovers, the gain margin is determined by that point which results in the gain margin with the smallest magnitude, in decibels.

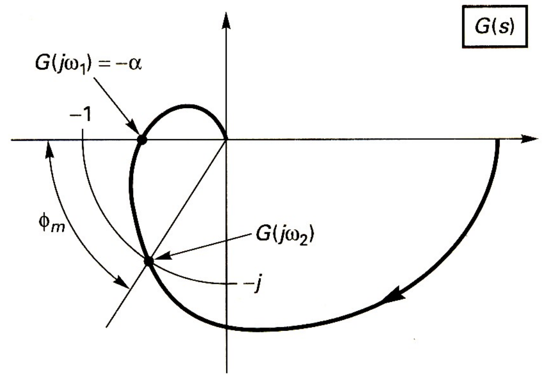

Figure 6.29: Relative stabilty margins.

6.4.2 The Phase Margin

The phase margin is the magnitude of the minimum angle by which the Nyquist diagram must be rotated in order to intersect the −1 point. The phase margin is indicated by the angle ϕ_m in Figure 6.29. The magnitude of the Nyquist diagram, $G(j\omega)$, is unity at the frequency that the phase margin occurs. This frequency is indicated as ω_2 in Figure 6.29, and thus $|G(j\omega)| = 1$. The phase margin is then

$$\phi_m = \underline{/G(j\omega_2)} - 180°$$

6.4.3 Relative stability and the Bode Plots

Although the gain and phase margins may be obtained directly from a Nyquist diagram, the are more often determined from a Bode diagram. For example, the Bode diagram for the system whose Nyquist diagram is shown in Figure 6.29 is given in Figure 6.30. The gain margin occurs at the frequency at which the phase angle of $G(j\omega)$ is −180○. This frequency is evident on the Bode diagram of Figure 6.30 and is labeled as ω_1 (we sometimes call this frequency the *phase*

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

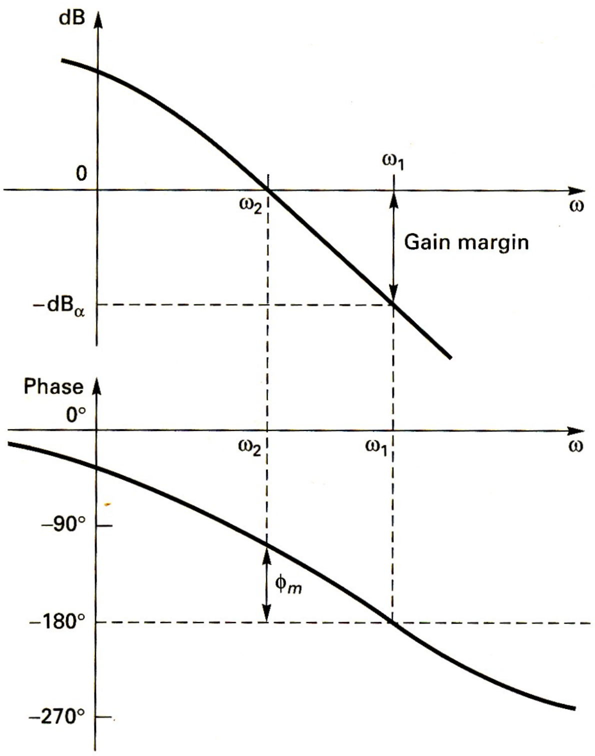

Figure 6.30: Relative stabilty margins on a Bode diagram.

crossover frequency). The gain margin is the reciprocal of the magnitude of $G(j\omega_1)$. Since the effect of taking the logarithm of a reciprocal of a number is

$$\log\left(\frac{1}{|G(j\omega_1)|}\right) = -\log|G(j\omega_1)|$$

and is expressed in decibels as the value dB_α in Figure 6.30. The phase margin occurs at the frequency ω_2 at which the magnitude of the open-loop gain is unity, or 0 dB (ω_2 is sometimes called the *gain crossover frequency*). The phase margin ϕ_m is the difference between the angle of $G(j\omega_2)$ and $-180°$, as shown in Figure 6.30.

In practical control system design, the straight line approximations for the Bode diagram are usually inadequate to determine the stability margins. Control engineers have found from experience that an 8 dB gain margin (a factor of 2.51) is usually adequate. In addition, a 50° phase margin is often adequate.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

6.5 Frequency Response Design

6.5.1 Gain compensation

6.5.2 Phase-Lead Compensation

As seen in chapter 5 a lead compensator has a transfer function of the form

$$D(s) = \frac{K(s+z)}{(s+p)}$$

with $|z| < |p|$. The frequency response of the compensator $D(s)$ is

$$D(j\omega) = \frac{K(j\omega+z)}{(j\omega+p)} = \frac{(Kz/p)[j(\omega/z)+1]}{[j(\omega/p)+1]} = \frac{K_1(1+j\omega\alpha\tau)}{(1+j\omega\tau)} \tag{6.11}$$

where $\tau = 1/p$, $p = \alpha z$, and $K_1 = K/\alpha$. We wish to determin the values of p and z such that certain design criteria will be satisfied for the closed loop system. The Bode diagram of the phase-lead compensator has the general form shown in Figure 6.31. We see that the phase-lead controller is a form of high-pass filter, in that the high frequencies are amplified relative to the low frequencies. The controller introduces gain at high frequencies, which in general is destabilizing. However, the positive phase angle of the controller tends to rotate the Nyquist diagram away from the -1 point and thus is stabilizing. Hence we must carefully choose the pole ans zero locations so that the stabilizing effect of the positive phase angle dominates.

The phase contributed by the lead compensator is given by

$$\phi = \tan^{-1}(\alpha\omega\tau) - \tan^{-1}(\omega\tau)$$

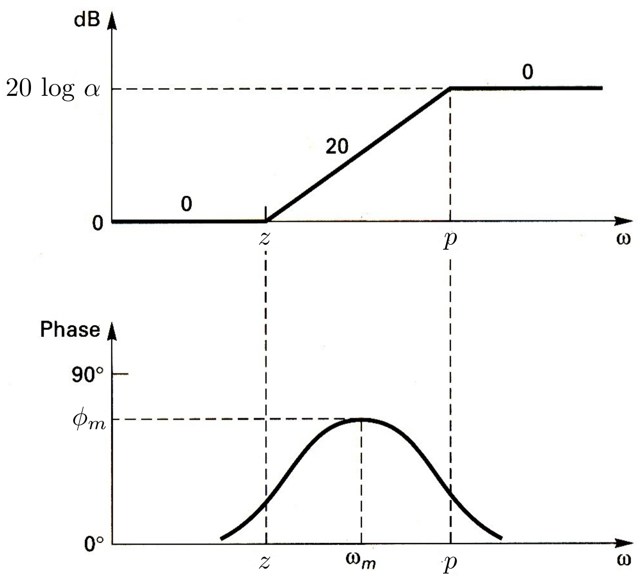

Figure 6.31: Bode diagram of phase-lead compensator.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

or equivalently

$$\phi = \tan^{-1}\left(\frac{\omega}{|z|}\right) - \tan^{-1}\left(\frac{\omega}{|p|}\right)$$

It can be shown that the frequency at which the phase is maximum is

$$\omega_m = \sqrt{|z||p|} = \frac{1}{\tau\sqrt{\alpha}}$$

and

$$\log \omega_m = \frac{1}{2}(\log|z| + \log|p|)$$

Note that the maximum phase lead occus halfway between the pole and zero frequencies on the logarithmic frequency scale, see Figure 6.31. For example, a lead compensator with a zero at $s = -2$ and a pole at $s = -10$, would yield a maximum phase lead at $\omega_m = \sqrt{2.10} = 4.47$ rad/sec To obtain an equation for the maximum phase-lead angle, ϕ_m, the frequency response in (6.11) is rewritten as

$$D(j\omega) = \frac{[1 + (\omega\tau)^2\alpha] + j(\alpha\omega\tau)}{[1 + (\omega\tau)^2\alpha]}$$

and hence the phase angle is

$$\phi = \tan^{-1}\frac{\alpha\omega\tau - \omega\tau}{1 + (\omega\tau)^2\alpha}$$

Then, substituting the frequency for the maximum phase angle, $\omega_m = 1/\tau\sqrt{\alpha}$, we have

$$\tan\phi_m = \frac{(\alpha/\sqrt{\alpha}) - (1/\sqrt{\alpha})}{1+1} = \frac{\alpha - 1}{2\sqrt{\alpha}}$$

Note that

$$\sin\phi_m = \frac{\alpha - 1}{\alpha + 1} \tag{6.12}$$

A plot of ϕ_m versus α is shown in Figure 6.32. The maximum value of the phase shift ϕ_m is then a function of α only.

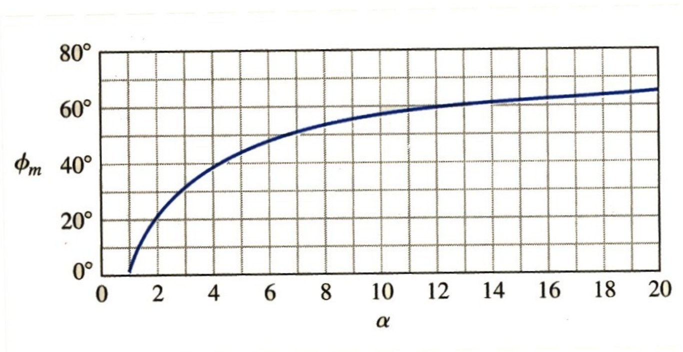

Figure 6.32: Maximum phase angle ϕ_m versus α for a phase-lead compensator.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Example 6.12 Consider an open-loop transfer function

$$G(s) = \frac{4}{s(s+1)}$$

We wish to design a compensator such that the closed loop system will satisfy the following requirements:

- velocity error constant, $K_v = 20$
- phase margin $= 50°$
- gain margin ≥ 10 db

We shall design a lead compensator $D(s)$ of the form

$$D(s) = K\frac{s+z}{s+p}$$

The first step in the design is to adjust the gain K to provide the required velocity error constant. First, we need to rewrite $D(s)$ as

$$D(s) = \frac{K_1[1+(s/z)]}{[1+(s/p)]}$$

where $K_1 = Kz/p$. Next

$$K_v = \lim_{s\to 0} sD(s)G(s) = s\frac{4K_1(1+s/z)}{s(s+2)(1+s/p)} = 2K_1 = 20$$

or $K_1 = 10$. Define

$$G_1(s) = K_1 G(s) = \frac{40}{s(s+2)}$$

We shall next plot the Bode diagram of $G_1(s)$, and is shown in Figure 6.33 From this plot, the phase margin is found to be 17°. The gain margin is ∞ dB. Since the specification calls for a phase margin of 50°, the additional phase lead necessary to satisfy the phase margin requirement is 33°.

Note that the addition of a lead compensator will modify both phase and magnitude. The magnitude curve in the Bode diagram is modified and the gain crossover frequency is shifted to the right. A small amount of safety (usually 10%) is added to compensate for this shift in the gain crossover frequency. Therefore, we assume that ϕ_m, the maximum phase lead required, is approximately 37°.

Next, using (6.12) we evaluate the value of α, we have

$$\frac{\alpha-1}{\alpha+1} = \sin 37°$$

and thus $\alpha = 4.023$.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

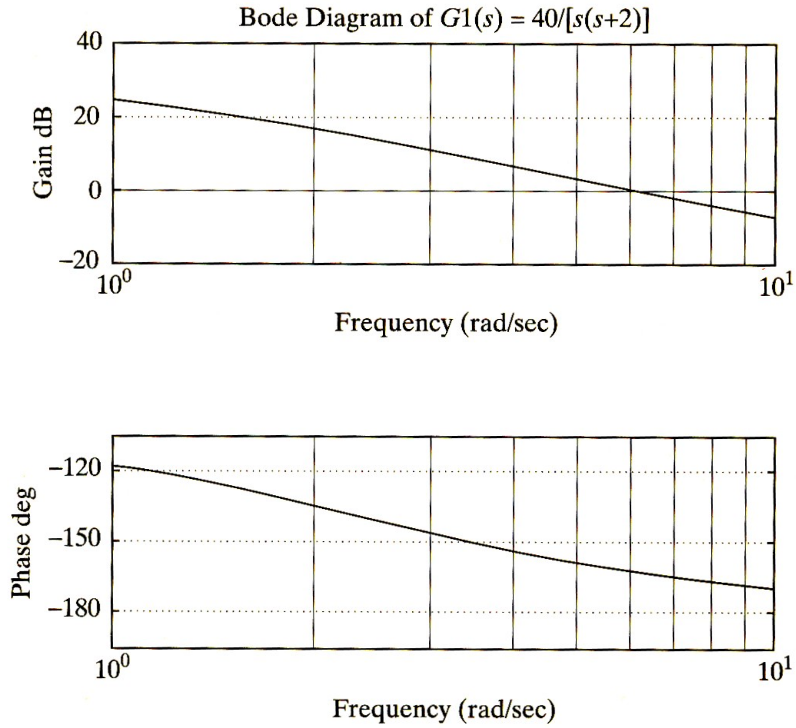

Figure 6.33: Bode diagram of $G_1(s) = \dfrac{40}{s(s+2)}$.