Coordinate Transformation Formula Sheet

Table with the Del operator in rectangular, cylindrical, and spherical coordinates			
Operation	**Cartesian coordinates (x,y,z)**	**Cylindrical coordinates (ρ,φ,z)**	**Spherical coordinates (r,θ, φ)**
Definition of coordinates	$\rho = \sqrt{x^2+y^2}$ $\phi = \arctan(y/x)$ $z = z$	$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$
	$r = \sqrt{x^2+y^2+z^2}$ $\theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$ $\phi = \arctan(y/x)$	$r = \sqrt{\rho^2+z^2}$ $\theta = \arctan(\rho/z)$ $\phi = \phi$	$\rho = r\sin(\theta)$ $\phi = \phi$ $z = r\cos(\theta)$
Definition of unit vectors	$\hat{\boldsymbol\rho} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$ $\hat{\phi} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \cos\phi\hat{\boldsymbol\rho} - \sin\phi\hat{\boldsymbol\phi}$ $\hat{\mathbf{y}} = \sin\phi\hat{\boldsymbol\rho} + \cos\phi\hat{\boldsymbol\phi}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \sin\theta\cos\phi\hat{\mathbf{r}} + \cos\theta\cos\phi\hat{\boldsymbol\theta} - \sin\phi\hat{\boldsymbol\phi}$ $\hat{\mathbf{y}} = \sin\theta\sin\phi\hat{\mathbf{r}} + \cos\theta\sin\phi\hat{\boldsymbol\theta} + \cos\phi\hat{\boldsymbol\phi}$ $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol\theta}$
	$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol\theta} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$ $\hat{\phi} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$	$\hat{\mathbf{r}} = \sin\theta\hat{\boldsymbol\rho} + \cos\theta\hat{\mathbf{z}}$ $\hat{\boldsymbol\theta} = \cos\theta\hat{\boldsymbol\rho} - \sin\theta\hat{\mathbf{z}}$ $\hat{\phi} = \hat{\phi}$	$\hat{\boldsymbol\rho} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol\theta}$ $\hat{\boldsymbol\phi} = \hat{\boldsymbol\phi}$ $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol\theta}$

A vector field **A**	$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$	$A_\rho\hat{\boldsymbol{\rho}} + A_\phi\hat{\boldsymbol{\phi}} + A_z\hat{\boldsymbol{z}}$	$A_r\hat{\boldsymbol{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$
Gradient ∇f	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$	$\frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$
Divergence $\nabla \cdot \mathbf{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$
Curl $\nabla \times \mathbf{A}$	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}}$ $+$ $\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}}$ $+$ $\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$	$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}}$ $+$ $\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}}$ $+$ $\frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\hat{\boldsymbol{z}}$	$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{\boldsymbol{r}}$ $+$ $\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\hat{\boldsymbol{\theta}}$ $+$ $\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}$
Laplace operator $\Delta f = \nabla^2 f$	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$
Vector Laplacian $\Delta\mathbf{A} = \nabla^2\mathbf{A}$	$\Delta A_x\hat{\mathbf{x}} + \Delta A_y\hat{\mathbf{y}} + \Delta A_z\hat{\mathbf{z}}$	$\left(\Delta A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi}\right)\hat{\boldsymbol{\rho}}$ $+$ $\left(\Delta A_\phi - \frac{A_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi}\right)\hat{\boldsymbol{\phi}}$ $+$ $(\Delta A_z)\hat{\boldsymbol{z}}$	$\left(\Delta A_r - \frac{2A_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)\hat{\boldsymbol{r}}$ $+$ $\left(\Delta A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}\right)\hat{\boldsymbol{\theta}}$ $+$ $\left(\Delta A_\phi - \frac{A_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}\right)\hat{\boldsymbol{\phi}}$
Differential displacement	$d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$	$d\mathbf{l} = d\rho\hat{\boldsymbol{\rho}} + \rho d\phi\hat{\boldsymbol{\phi}} + dz\hat{\boldsymbol{z}}$	$d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}$
Differential normal area	$d\mathbf{S} =$ $dy\,dz\,\hat{\mathbf{x}} +$ $dx\,dz\,\hat{\mathbf{y}} +$ $dx\,dy\,\hat{\mathbf{z}}$	$d\mathbf{S} =$ $\rho\,d\phi\,dz\,\hat{\boldsymbol{\rho}} +$ $d\rho\,dz\,\hat{\boldsymbol{\phi}} +$ $\rho\,d\rho d\phi\,\hat{\mathbf{z}}$	$d\mathbf{S} =$ $r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}} +$ $r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}} +$ $r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$

Differential volume	$dV = dx\,dy\,dz$	$dV = \rho\,d\rho\,d\phi\,dz$	$dV = r^2 \sin\theta\,dr\,d\theta\,d\phi$

Non-trivial calculation rules:

1. $\text{div grad } f = \nabla \cdot (\nabla f) = \nabla^2 f = \Delta f$
2. $\text{curl grad } f = \nabla \times (\nabla f) = \mathbf{0}$
3. $\text{div curl } \mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$
4. $\text{curl curl } \mathbf{A} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$
5. $\Delta f g = f \Delta g + 2 \nabla f \cdot \nabla g + g \Delta f$

Vector Transformations in Rectangular, Cylindrical, and Spherical Coordinates

From rectangular to cylindrical:

$$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

From cylindrical to rectangular:

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix}$$

From rectangular to spherical:

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

From spherical to rectangular:

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}$$

From cylindrical to spherical:

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix}$$

From spherical to cylindrical:

$$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}$$

Integration Formulas

1. $$\int x^a dx = \frac{x^{a+1}}{a+1}$$

2. $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(\sqrt{x^2 + a^2} + x\right)$$

3. $$\int \frac{dx}{\left(ax^2 + bx + c\right)^{3/2}} = \frac{2\left(2ax + b\right)}{\left(4ac - b^2\right)\left(ax^2 + bx + c\right)^{1/2}}$$

Table of Integrals*

Basic Forms

$$\int x^n dx = \frac{1}{n+1}x^{n+1} \tag{1}$$

$$\int \frac{1}{x} dx = \ln x \tag{2}$$

$$\int u dv = uv - \int v du \tag{3}$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| \tag{4}$$

Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a} \tag{5}$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1} + c, n \neq -1 \tag{6}$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} \tag{7}$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x \tag{8}$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \tag{9}$$

$$\int \frac{x}{a^2+x^2} dx = \frac{1}{2} \ln |a^2+x^2| \tag{10}$$

$$\int \frac{x^2}{a^2+x^2} dx = x - a \tan^{-1} \frac{x}{a} \tag{11}$$

$$\int \frac{x^3}{a^2+x^2} dx = \frac{1}{2}x^2 - \frac{1}{2}a^2 \ln |a^2+x^2| \tag{12}$$

$$\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} \tag{13}$$

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \frac{a+x}{b+x},\ a \neq b \tag{14}$$

$$\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln |a+x| \tag{15}$$

$$\int \frac{x}{ax^2+bx+c} dx = \frac{1}{2a} \ln |ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} \tag{16}$$

Integrals with Roots

$$\int \sqrt{x-a} dx = \frac{2}{3}(x-a)^{3/2} \tag{17}$$

$$\int \frac{1}{\sqrt{x \pm a}} dx = 2\sqrt{x \pm a} \tag{18}$$

$$\int \frac{1}{\sqrt{a-x}} dx = -2\sqrt{a-x} \tag{19}$$

$$\int x\sqrt{x-a} dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2} \tag{20}$$

$$\int \sqrt{ax+b} dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b} \tag{21}$$

$$\int (ax+b)^{3/2} dx = \frac{2}{5a}(ax+b)^{5/2} \tag{22}$$

$$\int \frac{x}{\sqrt{x \pm a}} dx = \frac{2}{3}(x \mp 2a)\sqrt{x \pm a} \tag{23}$$

$$\int \sqrt{\frac{x}{a-x}} dx = -\sqrt{x(a-x)} - a \tan^{-1} \frac{\sqrt{x(a-x)}}{x-a} \tag{24}$$

$$\int \sqrt{\frac{x}{a+x}} dx = \sqrt{x(a+x)} - a \ln \left[\sqrt{x} + \sqrt{x+a}\right] \tag{25}$$

$$\int x\sqrt{ax+b} dx = \frac{2}{15a^2}(-2b^2+abx+3a^2x^2)\sqrt{ax+b} \tag{26}$$

$$\int \sqrt{x(ax+b)} dx = \frac{1}{4a^{3/2}} \Big[(2ax+b)\sqrt{ax(ax+b)} - b^2 \ln \left|a\sqrt{x} + \sqrt{a(ax+b)}\right|\Big] \tag{27}$$

$$\int \sqrt{x^3(ax+b)} dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right] \sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}} \ln \left|a\sqrt{x} + \sqrt{a(ax+b)}\right| \tag{28}$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln \left|x + \sqrt{x^2 \pm a^2}\right| \tag{29}$$

$$\int \sqrt{a^2-x^2} dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2 \tan^{-1} \frac{x}{\sqrt{a^2-x^2}} \tag{30}$$

$$\int x\sqrt{x^2 \pm a^2} dx = \frac{1}{3}\left(x^2 \pm a^2\right)^{3/2} \tag{31}$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left|x + \sqrt{x^2 \pm a^2}\right| \tag{32}$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} \tag{33}$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} \tag{34}$$

$$\int \frac{x}{\sqrt{a^2-x^2}} dx = -\sqrt{a^2-x^2} \tag{35}$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}a^2 \ln \left|x + \sqrt{x^2 \pm a^2}\right| \tag{36}$$

$$\int \sqrt{ax^2+bx+c} dx = \frac{b+2ax}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a^{3/2}} \ln \left|2ax+b+2\sqrt{a(ax^2+bx^{+}c)}\right| \tag{37}$$

$$\int x\sqrt{ax^2+bx+c} = \frac{1}{48a^{5/2}}\Big(2\sqrt{a}\sqrt{ax^2+bx+c} - \left(3b^2+2abx+8a(c+ax^2)\right) + 3(b^3-4abc) \ln \left|b+2ax+2\sqrt{a}\sqrt{ax^2+bx+x}\right|\Big) \tag{38}$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}} dx = \frac{1}{\sqrt{a}} \ln \left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| \tag{39}$$

$$\int \frac{x}{\sqrt{ax^2+bx+c}} dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}} \ln \left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| \tag{40}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} \tag{41}$$

Integrals with Logarithms

$$\int \ln ax dx = x \ln ax - x \tag{42}$$

$$\int \frac{\ln ax}{x} dx = \frac{1}{2}(\ln ax)^2 \tag{43}$$

$$\int \ln(ax+b) dx = \left(x + \frac{b}{a}\right) \ln(ax+b) - x, a \neq 0 \tag{44}$$

$$\int \ln(x^2+a^2)\ \mathrm{dx} = x \ln(x^2+a^2) + 2a \tan^{-1} \frac{x}{a} - 2x \tag{45}$$

$$\int \ln(x^2-a^2)\ \mathrm{dx} = x \ln(x^2-a^2) + a \ln \frac{x+a}{x-a} - 2x \tag{46}$$

$$\int \ln\left(ax^2+bx+c\right) dx = \frac{1}{a}\sqrt{4ac-b^2} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} - 2x + \left(\frac{b}{2a} + x\right) \ln\left(ax^2+bx+c\right) \tag{47}$$

$$\int x \ln(ax+b) dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2 - \frac{b^2}{a^2}\right) \ln(ax+b) \tag{48}$$

$$\int x \ln\left(a^2-b^2x^2\right) dx = -\frac{1}{2}x^2 + \frac{1}{2}\left(x^2 - \frac{a^2}{b^2}\right) \ln\left(a^2-b^2x^2\right) \tag{49}$$

Integrals with Exponentials

$$\int e^{ax} dx = \frac{1}{a}e^{ax} \tag{50}$$

$$\int \sqrt{x}e^{ax} dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}} \operatorname{erf}\left(i\sqrt{ax}\right), \text{ where } \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{51}$$

$$\int xe^x dx = (x-1)e^x \tag{52}$$

$$\int xe^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax} \tag{53}$$

$$\int x^2e^x dx = \left(x^2-2x+2\right) e^x \tag{54}$$

$$\int x^2e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) e^{ax} \tag{55}$$

$$\int x^3e^x dx = \left(x^3-3x^2+6x-6\right) e^x \tag{56}$$

$$\int x^n e^{ax}\ \mathrm{dx} = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax}\ \mathrm{dx} \tag{57}$$

$$\int x^n e^{ax}\ \mathrm{dx} = \frac{(-1)^n}{a^{n+1}} \Gamma[1+n, -ax], \text{ where } \Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\ \mathrm{dt} \tag{58}$$

$$\int e^{ax^2}\ \mathrm{dx} = -\frac{i\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}\left(ix\sqrt{a}\right) \tag{59}$$

$$\int e^{-ax^2}\ \mathrm{dx} = \frac{\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}\left(x\sqrt{a}\right) \tag{60}$$

$$\int xe^{-ax^2}\ \mathrm{dx} = -\frac{1}{2a}e^{-ax^2} \tag{61}$$

$$\int x^2e^{-ax^2}\ \mathrm{dx} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \operatorname{erf}(x\sqrt{a}) - \frac{x}{2a}e^{-ax^2} \tag{62}$$

Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a}\cos ax \tag{63}$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \tag{64}$$

$$\int \sin^n ax dx = -\frac{1}{a}\cos ax \;\; {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right] \tag{65}$$

$$\int \sin^3 ax dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a} \tag{66}$$

$$\int \cos ax dx = \frac{1}{a}\sin ax \tag{67}$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \tag{68}$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)}\cos^{1+p} ax \times {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right] \tag{69}$$

$$\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a} \tag{70}$$

$$\int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b \tag{71}$$

$$\int \sin^2 ax \cos bx dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} \tag{72}$$

$$\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x \tag{73}$$

$$\int \cos^2 ax \sin bx dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)} \tag{74}$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a}\cos^3 ax \tag{75}$$

$$\int \sin^2 ax \cos^2 bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} \tag{76}$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a} \tag{77}$$

$$\int \tan ax dx = -\frac{1}{a}\ln \cos ax \tag{78}$$

$$\int \tan^2 ax dx = -x + \frac{1}{a}\tan ax \tag{79}$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right) \tag{80}$$

$$\int \tan^3 ax dx = \frac{1}{a}\ln \cos ax + \frac{1}{2a}\sec^2 ax \tag{81}$$

$$\int \sec x dx = \ln|\sec x + \tan x| = 2\tanh^{-1}\left(\tan\frac{x}{2}\right) \tag{82}$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax \tag{83}$$

$$\int \sec^3 x \,\mathrm{dx} = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| \tag{84}$$

$$\int \sec x \tan x dx = \sec x \tag{85}$$

$$\int \sec^2 x \tan x dx = \frac{1}{2}\sec^2 x \tag{86}$$

$$\int \sec^n x \tan x dx = \frac{1}{n}\sec^n x, n \neq 0 \tag{87}$$

$$\int \csc x dx = \ln\left|\tan\frac{x}{2}\right| = \ln|\csc x - \cot x| + C \tag{88}$$

$$\int \csc^2 ax dx = -\frac{1}{a}\cot ax \tag{89}$$

$$\int \csc^3 x dx = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x| \tag{90}$$

$$\int \csc^n x \cot x dx = -\frac{1}{n}\csc^n x, n \neq 0 \tag{91}$$

$$\int \sec x \csc x dx = \ln|\tan x| \tag{92}$$

Products of Trigonometric Functions and Monomials

$$\int x \cos x dx = \cos x + x \sin x \tag{93}$$

$$\int x \cos ax dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax \tag{94}$$

$$\int x^2 \cos x dx = 2x\cos x + \left(x^2 - 2\right)\sin x \tag{95}$$

$$\int x^2 \cos ax dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin ax \tag{96}$$

$$\int x^n cosx dx = -\frac{1}{2}(i)^{n+1}\left[\Gamma(n+1, -ix) + (-1)^n\Gamma(n+1, ix)\right] \tag{97}$$

$$\int x^n cosax dx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n\Gamma(n+1, -iax) - \Gamma(n+1, ixa)\right] \tag{98}$$

$$\int x \sin x dx = -x\cos x + \sin x \tag{99}$$

$$\int x \sin ax dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2} \tag{100}$$

$$\int x^2 \sin x dx = \left(2 - x^2\right)\cos x + 2x\sin x \tag{101}$$

$$\int x^2 \sin ax dx = \frac{2 - a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2} \tag{102}$$

$$\int x^n \sin x dx = -\frac{1}{2}(i)^n\left[\Gamma(n+1, -ix) - (-1)^n\Gamma(n+1, -ix)\right] \tag{103}$$

Products of Trigonometric Functions and Exponentials

$$\int e^x \sin x dx = \frac{1}{2}e^x(\sin x - \cos x) \tag{104}$$

$$\int e^{bx} \sin ax dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax) \tag{105}$$

$$\int e^x \cos x dx = \frac{1}{2}e^x(\sin x + \cos x) \tag{106}$$

$$\int e^{bx} \cos ax dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax) \tag{107}$$

$$\int xe^x \sin x dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x) \tag{108}$$

$$\int xe^x \cos x dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x) \tag{109}$$

Integrals of Hyperbolic Functions

$$\int \cosh ax dx = \frac{1}{a}\sinh ax \tag{110}$$

$$\int e^{ax} \cosh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[a\cosh bx - b\sinh bx] & a \neq b \\ \dfrac{e^{2ax}}{4a} + \dfrac{x}{2} & a = b \end{cases} \tag{111}$$

$$\int \sinh ax dx = \frac{1}{a}\cosh ax \tag{112}$$

$$\int e^{ax} \sinh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[-b\cosh bx + a\sinh bx] & a \neq b \\ \dfrac{e^{2ax}}{4a} - \dfrac{x}{2} & a = b \end{cases} \tag{113}$$

$$\int e^{ax} \tanh bx dx = \begin{cases} \dfrac{e^{(a+2b)x}}{(a+2b)}\,{}_2F_1\left[1 + \dfrac{a}{2b}, 1, 2 + \dfrac{a}{2b}, -e^{2bx}\right] \\ \quad -\dfrac{1}{a}e^{ax}\,{}_2F_1\left[\dfrac{a}{2b}, 1, 1E, -e^{2bx}\right] & a \neq b \\ \dfrac{e^{ax} - 2\tan^{-1}[e^{ax}]}{a} & a = b \end{cases} \tag{114}$$

$$\int \tanh ax \, dx = \frac{1}{a}\ln\cosh ax \tag{115}$$

$$\int \cos ax \cosh bx dx = \frac{1}{a^2+b^2}\left[a\sin ax\cosh bx + b\cos ax\sinh bx\right] \tag{116}$$

$$\int \cos ax \sinh bx dx = \frac{1}{a^2+b^2}\left[b\cos ax\cosh bx + a\sin ax\sinh bx\right] \tag{117}$$

$$\int \sin ax \cosh bx dx = \frac{1}{a^2+b^2}\left[-a\cos ax\cosh bx + b\sin ax\sinh bx\right] \tag{118}$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2+b^2}\left[b\cosh bx\sin ax - a\cos ax\sinh bx\right] \tag{119}$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a}\left[-2ax + \sinh 2ax\right] \tag{120}$$

$$\int \sinh ax \cosh bx dx = \frac{1}{b^2-a^2}\left[b\cosh bx\sinh ax - a\cosh ax\sinh bx\right] \tag{121}$$