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Magnetic Forces and Torques

Magnetic Force on a Moving Charge

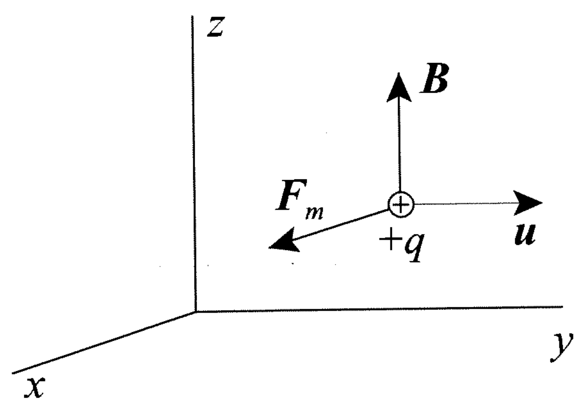
Electric charges moving in a magnetic field experience a force due to the magnetic field. Given a point charge q moving with velocity $\boldsymbol{u}$ in a magnetic flux density $\boldsymbol{B}$, the vector magnetic force $\boldsymbol{F}_m$ on the charge is given by

$$\boldsymbol{F}_m = q(\boldsymbol{u} \times \boldsymbol{B}) \qquad \text{(N)}$$

Note that the force is normal to the plane containing the velocity vector and the magnetic flux density vector. Also note that the force is zero if the charge is stationary ($\boldsymbol{u}$=0).

Example (Force on a point charge moving in a magnetic field)

Determine the vector magnetic force on a point charge $+q$ moving at a uniform velocity $\boldsymbol{u} = u_o\hat{\boldsymbol{y}}$ in a uniform magnetic flux density defined by $\boldsymbol{B} = B_o\hat{\boldsymbol{z}}$.



$$\begin{aligned}\boldsymbol{F}_m &= q(\boldsymbol{u} \times \boldsymbol{B}) \\ &= q[(u_o\hat{\boldsymbol{y}}) \times (B_o\hat{\boldsymbol{z}})] \\ &= q u_o B_o \hat{\boldsymbol{x}}\end{aligned}$$

Given a charge moving in an electric field and a magnetic field, the total force on the charge is the superposition of electric and magnetic forces. This total force equation is known as the *Lorentz force equation.* The vector force component due to the electric field ($\boldsymbol{F}_e$) is given by $\boldsymbol{F}_e = q\boldsymbol{E}$. The total vector force on the point charge is (Lorentz force - $\boldsymbol{F}$) is

$$\boldsymbol{F} = \boldsymbol{F}_e + \boldsymbol{F}_m = q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \qquad \text{(N)} \qquad \left(\begin{array}{c}\text{Lorentz force}\\ \text{equation}\end{array}\right)$$

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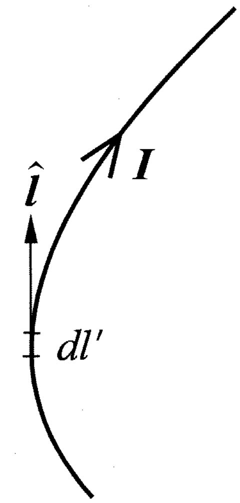
Magnetic Forces on Current-Carrying Conductors

Given that charge moving in a magnetic field experiences a force, a current carrying conductor in a magnetic field also experiences a force. The current carrying conductor (modeled as a line current) can be subdivided into differential current elements (differential lengths).

The charge-velocity product for a moving point charge can be related to an equivalent differential length of line current.



$$q\boldsymbol{u} = q\frac{dl'}{t}\hat{\boldsymbol{l}} = \frac{q}{t}dl'\hat{\boldsymbol{l}} = I\hat{\boldsymbol{l}}dl' = \boldsymbol{I}dl' \qquad \text{(A-m)}$$

The equivalence of the moving point charge and the differential length of line current yields the equivalent magnetic force equation.

$$\boldsymbol{F}_m = q(\boldsymbol{u}\times\boldsymbol{B}) \quad \Leftrightarrow \quad d\boldsymbol{F}_m = dl'(\boldsymbol{I}\times\boldsymbol{B})$$

The total force on the current carrying conductor is found by summing the forces on all of the differentials elements of current (integrating along the length of the conductor).

$$\boldsymbol{F}_m = \int_L (\boldsymbol{I}\times\boldsymbol{B})\,dl'$$

Given a steady current, the magnitude of the current is constant along the length of the conductor so that the magnetic force can be written as

$$\boldsymbol{F}_m = I\int_L (\boldsymbol{dl'}\times\boldsymbol{B})$$

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Torque on a Current Loop

Given the change in current directions around a closed current loop, the magnetic forces on different portions of the loop vary in direction. Using the Lorentz force equation, we can show that the net force on a simple circular or rectangular loop is a torque which forces the loop to align its *magnetic moment* with the applied magnetic field.

Consider the rectangular current loop shown below. The loop lies in the *x-y* plane and carries a DC current *I*. The loop lies in a uniform magnetic flux density $\boldsymbol{B}$ given by

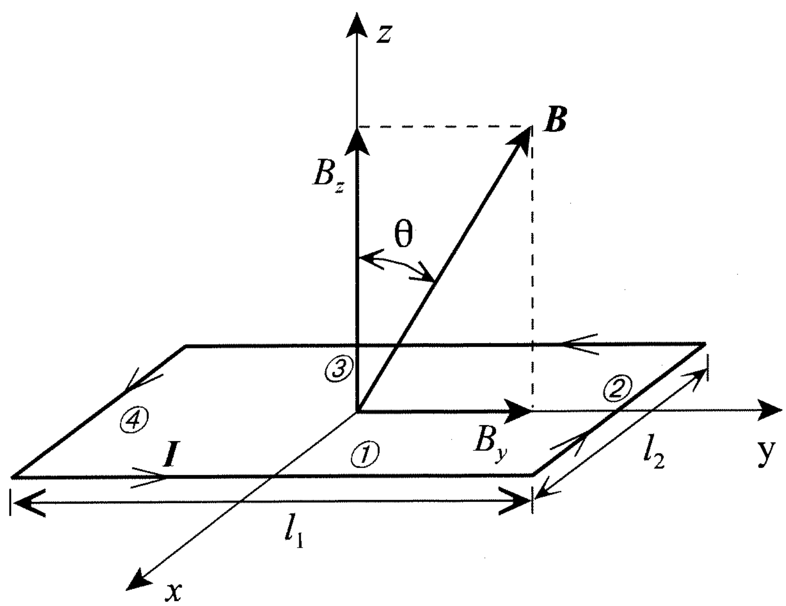
$$\boldsymbol{B} = B_y\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}}$$

The loop consists of four distinct vector current segments.

$$\boldsymbol{I}_1 = I\hat{\boldsymbol{y}} \qquad \boldsymbol{I}_2 = -I\hat{\boldsymbol{x}} \qquad \boldsymbol{I}_3 = -I\hat{\boldsymbol{y}} \qquad \boldsymbol{I}_4 = I\hat{\boldsymbol{x}}$$

Given a uniform flux density and a DC current along straight current segments, the magnetic force on each conductor segment can be simplified to the following equation.

$$\boldsymbol{F}_m = \int_L (\boldsymbol{I}\times\boldsymbol{B})\,dl' = (\boldsymbol{I}\times\boldsymbol{B})\int_L dl' = (\boldsymbol{I}\times\boldsymbol{B})L$$



The forces on the current segments can be determined for each component of the magnetic flux density.

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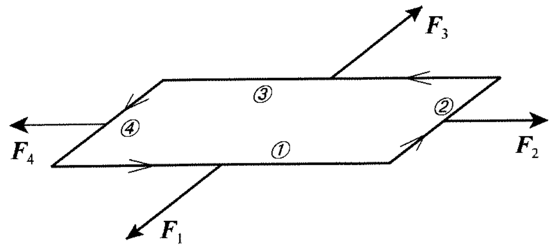
<u>Forces due to B_z</u> (net force = 0)

$$\boldsymbol{F}_1 = (I\hat{\boldsymbol{y}} \times B_z\hat{\boldsymbol{z}})l_1 = IB_z l_1 \hat{\boldsymbol{x}}$$

$$\boldsymbol{F}_2 = (-I\hat{\boldsymbol{x}} \times B_z\hat{\boldsymbol{z}})l_2 = IB_z l_2 \hat{\boldsymbol{y}}$$

$$\boldsymbol{F}_3 = (-I\hat{\boldsymbol{y}} \times B_z\hat{\boldsymbol{z}})l_1 = -IB_z l_1 \hat{\boldsymbol{x}}$$

$$\boldsymbol{F}_4 = (I\hat{\boldsymbol{x}} \times B_z\hat{\boldsymbol{z}})l_2 = -IB_z l_2 \hat{\boldsymbol{y}}$$



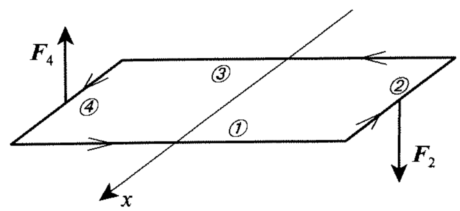
<u>Forces due to B_y</u> (net force = vector torque $\boldsymbol{T}$)

$$\boldsymbol{F}_1 = (I\hat{\boldsymbol{y}} \times B_y\hat{\boldsymbol{y}})l_1 = \boldsymbol{0}$$

$$\boldsymbol{F}_2 = (-I\hat{\boldsymbol{x}} \times B_y\hat{\boldsymbol{y}})l_2 = -IB_y l_2 \hat{\boldsymbol{z}}$$

$$\boldsymbol{F}_3 = (-I\hat{\boldsymbol{y}} \times B_y\hat{\boldsymbol{y}})l_1 = \boldsymbol{0}$$

$$\boldsymbol{F}_4 = (I\hat{\boldsymbol{x}} \times B_y\hat{\boldsymbol{y}})l_2 = IB_y l_2 \hat{\boldsymbol{z}}$$



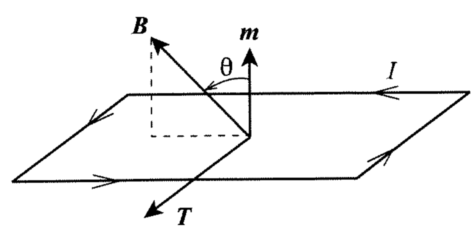
The vector torque on the loop is defined in terms of the force magnitude ($IB_y l_2$), the torque moment arm distance ($l_1/2$), and the torque direction (defined by the right hand rule):

$$\boldsymbol{T} = 2(IB_y l_2)\left(\frac{l_1}{2}\right)(-\hat{\boldsymbol{x}}) = -IB_y l_1 l_2 \hat{\boldsymbol{x}} = -IAB_y \hat{\boldsymbol{x}}$$

where $A = l_1 l_2$ is the loop area. The vector torque can be written in a general way in terms of the *vector magnetic moment* ($\boldsymbol{m}$) of the loop.

$$\boldsymbol{m} = IA\hat{\boldsymbol{n}} \qquad \text{(vector magnetic moment)}$$

where $\hat{\boldsymbol{n}}$ is the unit normal to the loop (defined by the right hand rule as applied to the current direction). The vector torque in terms of the magnetic moment is



$$\boldsymbol{T} = \boldsymbol{m} \times \boldsymbol{B} = mB\sin\theta\, \hat{\boldsymbol{a}}_T = IAB\sin\theta\, \hat{\boldsymbol{a}}_T$$

Note that the torque on the loop tends to align the loop magnetic moment with the direction of the applied magnetic field.

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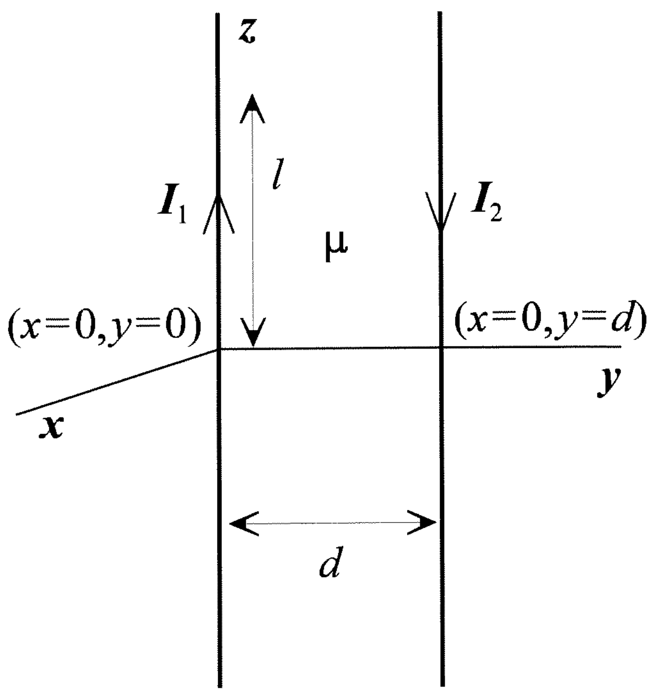
Forces Between Current Carrying Conductors

Given that any current carrying conductor produces a magnetic field, when two current carrying conductors are brought into close proximity, each conductor lies in the magnetic field of the other conductor. Therefore, both current carrying conductors exert a force on the opposite conductor.

Example (Force between line currents)

Determine the force/unit length on a line current I_1 due to the magnetic flux produced by a parallel line current I_2 (separation distance = d) flowing in the opposite direction.



The magnetic flux density at the location of I_1 due to the current I_2 is

$$\boldsymbol{B}_2(x=0, y=0) = \frac{\mu I_2}{2\pi d}(-\hat{\boldsymbol{x}})$$

The force on a length l of current I_1 due to the flux produced by I_2 is

$$\boldsymbol{F}_1 = \int_0^l (\boldsymbol{I}_1 \times \boldsymbol{B}_2)\, dz'$$

$$= \int_0^l \left[(I_1 \hat{\boldsymbol{z}}) \times \left(\frac{\mu I_2}{2\pi d}(-\hat{\boldsymbol{x}}) \right) \right] dz'$$

$$= \frac{\mu I_1 I_2}{2\pi d}(-\hat{\boldsymbol{y}}) \int_0^l dz'$$

$$= -\frac{\mu I_1 I_2 l}{2\pi d}\hat{\boldsymbol{y}}$$

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The force per unit length on the current I_1 is

$$\frac{\boldsymbol{F}_1}{l} = -\frac{\mu I_1 I_2}{2\pi d}\hat{\boldsymbol{y}}$$

The force on a length l of current I_2 due to the flux produced by I_1 is

$$\boldsymbol{F}_2 = \int_0^l (\boldsymbol{I}_2 \times \boldsymbol{B}_1)\, dz'$$

$$= \int_0^l \left[(-I_2\hat{\boldsymbol{z}}) \times \left(\frac{\mu I_1}{2\pi d}(-\hat{\boldsymbol{x}}) \right) \right] dz'$$

$$= \frac{\mu I_1 I_2}{2\pi d}(\hat{\boldsymbol{y}}) \int_0^l dz'$$

$$= \frac{\mu I_1 I_2 l}{2\pi d}\hat{\boldsymbol{y}}$$

The force per unit length on the current I_2 is

$$\frac{\boldsymbol{F}_2}{l} = \frac{\mu I_1 I_2}{2\pi d}\hat{\boldsymbol{y}}$$

Note that the currents repel each other given the currents flowing in opposite directions. If the currents flow in the same direction, they attract each another.

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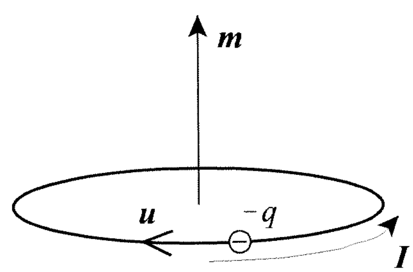
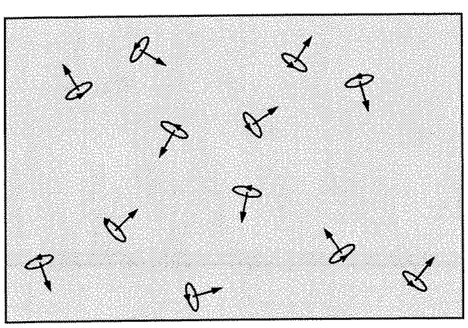
Magnetization and Magnetic Materials

Magnetization

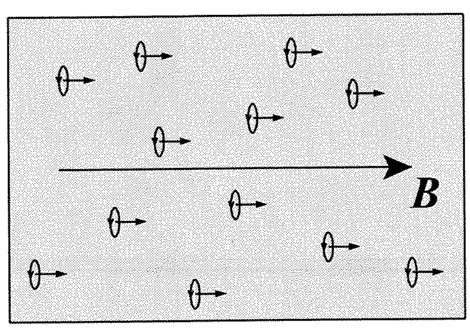
Just as dielectric materials are *polarized* under the influence of an applied electric field, certain materials can be *magnetized* under the influence of an applied magnetic field. Magnetization for magnetic fields is the dual process to polarization for electric fields. The magnetization process may be defined using the magnetic moments of the electron orbits within the atoms of the material. Each orbiting electron can be viewed as a small current loop with an associated magnetic moment.



An unmagnetized material can be characterized by a random distribution of the magnetic moments associated with the electron orbits. These randomly oriented magnetic moments produce magnetic field components that tend to cancel one another (net $\boldsymbol{H}=0$). Under the influence of an applied magnetic field, many of the current loops align their magnetic moments in the direction of the applied magnetic field.



Unmagnetized
(random moments)



Magnetized
(aligned moments)

If most of the magnetic moments stay aligned after the applied magnetic field is removed, a *permanent magnet* is formed.

A small current loop is commonly referred to as a *magnetic dipole*. The far field electric field of the electric dipole (produced during polarization) is functionally the same as the magnetic field of the magnetic dipole (produced during magnetization).

$$\boldsymbol{E} = \frac{p}{4\pi\epsilon r^3}\left[2\cos\theta\,\boldsymbol{a}_r + \sin\theta\,\boldsymbol{a}_\theta\right] \qquad \text{(electric dipole)}$$

$$\boldsymbol{B} = \frac{\mu m}{4\pi r^3}\left[2\cos\theta\,\boldsymbol{a}_r + \sin\theta\,\boldsymbol{a}_\theta\right] \qquad \text{(magnetic dipole)}$$

The preceding equations assume the dipole is centered at the coordinate origin and oriented with its dipole moment along the z-axis.

The parameters associated with the magnetization process are duals to those of the polarization process. The magnetization vector $\boldsymbol{M}$ is the dual of the polarization vector $\boldsymbol{P}$ and is defined as the magnetic dipole moment per unit volume.

Magnetization	Polarization
$M = \frac{m}{v} \left(\frac{\text{magnetic moment}}{\text{unit volume}}\right)$	$P = \frac{ql}{v} \left(\frac{\text{dipole moment}}{\text{unit volume}}\right)$
$\boldsymbol{B} = \mu_o(\boldsymbol{H}+\boldsymbol{M}) = \mu_o\mu_r\boldsymbol{H} = \mu\boldsymbol{H}$	$\boldsymbol{D} = \epsilon_o\boldsymbol{E} + \boldsymbol{P} = \epsilon_o\epsilon_r\boldsymbol{E} = \epsilon\boldsymbol{E}$
$\mu_r = 1 + \frac{M}{H} = 1 + \chi_m$	$\epsilon_r = 1 + \frac{P}{\epsilon_o E} = 1 + \chi_e$
$\boldsymbol{M} = \chi_m\boldsymbol{H}$	$\boldsymbol{P} = \chi_e\epsilon_o\boldsymbol{E}$

Note that the magnetic susceptibility χ_m is defined somewhat differently than the electric susceptibility χ_e. However, just as the electric susceptibility and relative permittivity are a measure of how much polarization occurs in the material, the magnetic susceptibility and relative permeability are a measure of how much magnetization occurs in the material.

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Magnetic Materials

Magnetic materials can be classified based on the magnitude of the relative permeability. Materials with a relative permeability of just under one (a small negative magnetic susceptibility) are defined as *diamagnetic*. In diamagnetic materials, the magnetic moments due to electron orbits and electron spin are very nearly equal and opposite such that they cancel each other. Thus, in diamagnetic materials, the response to an applied magnetic field is a slight magnetic field in the opposite direction. *Superconductors* exhibit perfect diamagnetism ($\chi_m = -1$) at temperatures near absolute zero such that magnetic fields cannot exist inside these materials.

Materials with a relative permeability of just greater than one are defined as *paramagnetic*. In paramagnetic materials, the magnetic moments due to electron orbit and spin are unequal, resulting in a small positive magnetic susceptibility. Magnetization is not significant in paramagnetic materials. Both diamagnetic and paramagnetic materials are typically linear media.

Materials with a relative permeability much greater than one are defined as *ferromagnetic*. Ferromagnetic materials are always nonlinear. As such, these materials cannot be described by a single value of relative permeability. If a single number is given for the relative permeability of any ferromagnetic material, this number represents an average value of μ_r.

Diamagnetic	$\mu_r < 1$	linear	$\boldsymbol{B} = \mu \boldsymbol{H}$
Paramagnetic	$\mu_r > 1$	linear	
Ferromagnetic	$\mu_r \gg 1$	nonlinear	$\boldsymbol{B} = \mu(H)\boldsymbol{H}$

Ferromagnetic materials lose their ferromagnetic properties at very high temperatures (above a temperature known as the *Curie temperature*).

The characteristics of ferromagnetic materials are typically presented using the *B-H* curve, a plot of the magnetic flux density *B* in the material due to a given applied magnetic field *H*.

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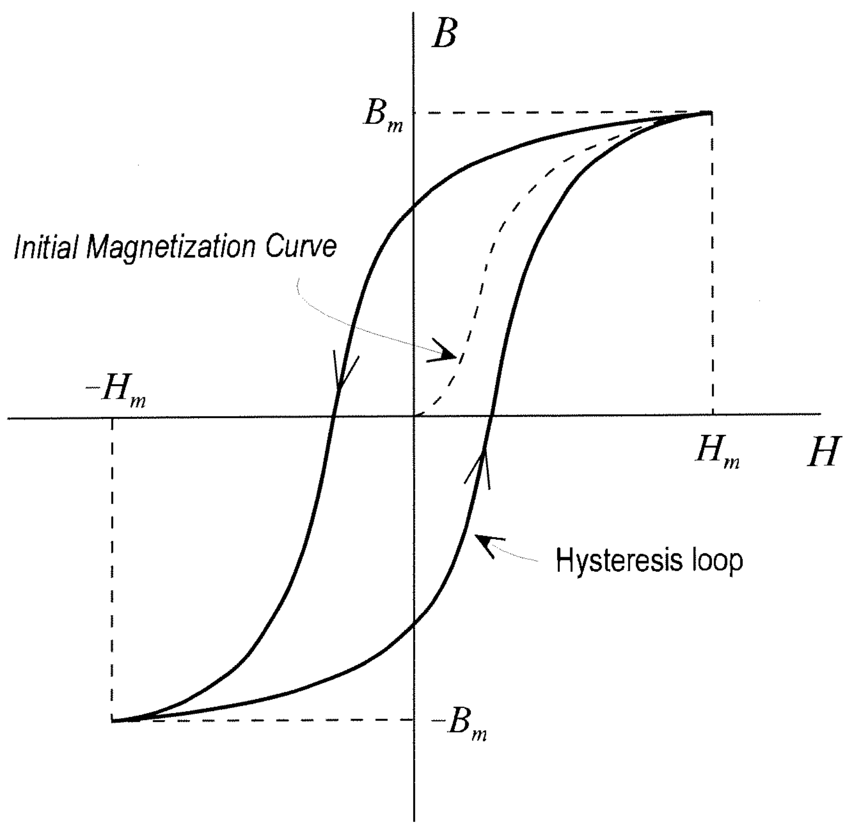
The *B-H* curve shows the *initial magnetization curve* along with a curve known as a *hysteresis loop*. The initial magnetization curve shows the magnetic flux density that would result when an increasing magnetic field is applied to an initially unmagnetized material. An unmagnetized material is defined by the $B=H=0$ point on the *B-H* curve (no net magnetic flux given no applied field). As the magnetic field increases, at some point, all of the magnetic moments (current loops) within the material align themselves with the applied field and the magnetic flux density saturates (B_m). If the magnetic field is then cycled between the saturation magnetic field value in the forward and reverse directions ($\pm H_m$), the hysteresis loop results. The response of the material to any applied field depends on the initial state of the material magnetization at that instant.



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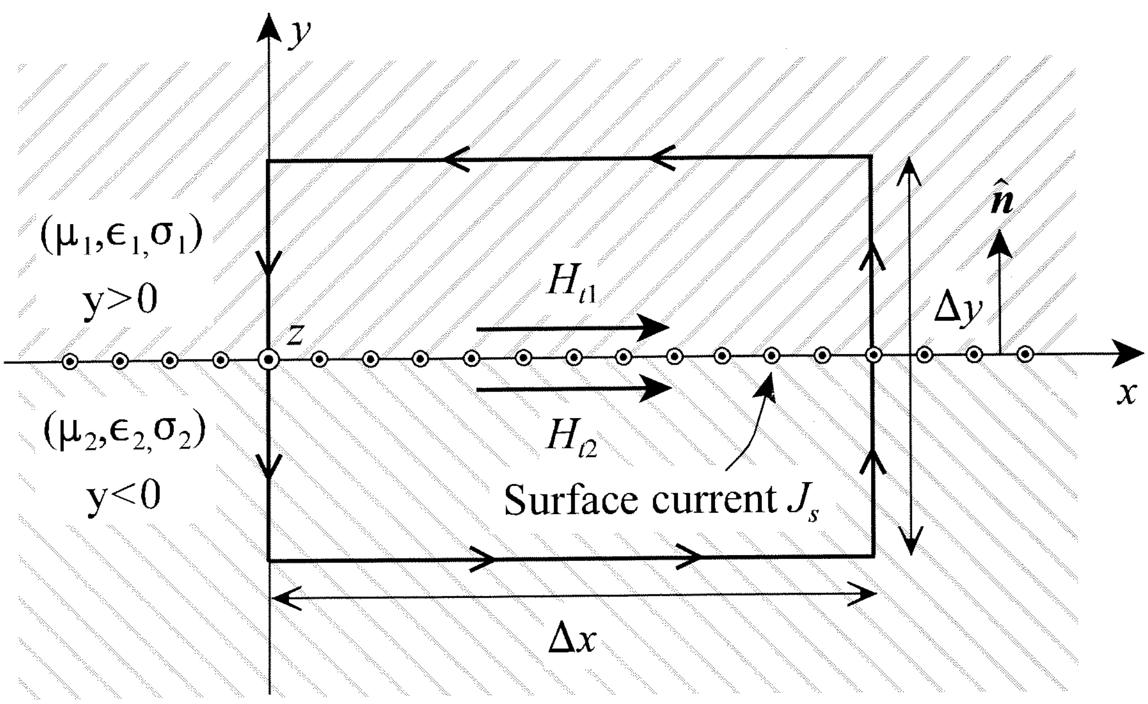
Magnetic Boundary Conditions

The fundamental boundary conditions involving magnetic fields relate the tangential components of magnetic field and the normal components of magnetic flux density on either side of the media interface. The same techniques used to determine the electric field boundary conditions can be used to determine the magnetic field boundary conditions. The tangential magnetic field boundary condition is found by applying Ampere's law on a path that straddles the media interface while the normal magnetic flux boundary condition is found by applying Gauss's law for magnetic fields to a volume straddling the media interface. The resulting boundary conditions are shown below.

Tangential Magnetic Field



$$\hat{\boldsymbol{n}} \times [\boldsymbol{H}_1 - \boldsymbol{H}_2] = \boldsymbol{J}_s$$

Vector boundary condition relating the magnetic field and surface current at a media interface.

where $\hat{\boldsymbol{n}}$ is a unit normal to the interface pointing into region 1.

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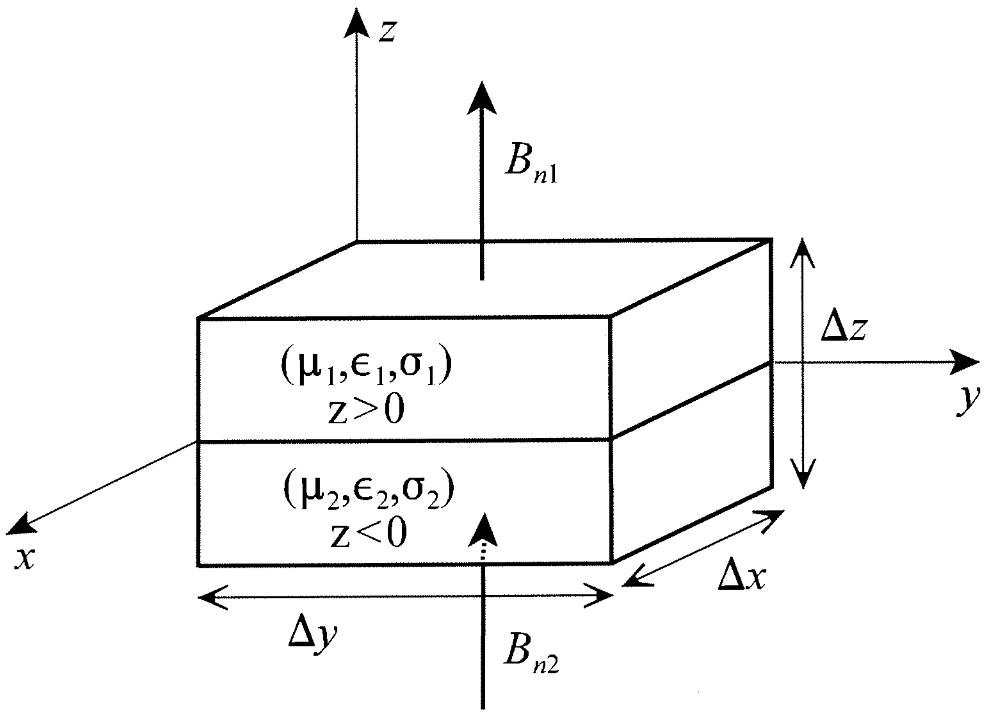
Normal Magnetic Flux Density



$$B_{n1} = B_{n2}$$

The normal components of magnetic flux density are continuous across a media interface.

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Inductors and Inductance

An *inductor* is an energy storage device that stores energy in an magnetic field. A inductor typically consists of some configuration of conductor coils (an efficient way of concentrating the magnetic field). Yet, even straight conductors contain inductance. The parameters that define inductors and inductance can be defined as parallel quantities to those of capacitors and capacitance.

<u>*Inductor*</u>	<u>*Capacitor*</u>
Stores energy in a magnetic field	Stores energy in an electric field
Inductance Definition $L \equiv \frac{\Lambda}{I}$	Capacitance Definition $C \equiv \frac{Q}{V}$
L = Inductance (H) Λ = Flux linkage (Wb) I = Current (A)	C = Capacitance (F) Q = Charge (C) V = Voltage (V)
$\Lambda = N\iint \boldsymbol{B}\cdot d\boldsymbol{s} = N\psi_m$	$Q = \iint \boldsymbol{D}\cdot d\boldsymbol{s} = \psi$
$W_m = \frac{1}{2}LI^2 = \frac{1}{2}\Lambda I = \frac{1}{2}\frac{\Lambda^2}{L}$	$W_e = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{C}$
$w_m = \frac{1}{2}\mu H^2$	$w_e = \frac{1}{2}\epsilon E^2$
$W_m = \iiint w_m dv = \frac{1}{2}\iiint \mu H^2 dv = \frac{1}{2}\iiint \boldsymbol{B}\cdot\boldsymbol{H}\, dv$	$W_e = \iiint w_e dv = \frac{1}{2}\iiint \epsilon E^2 dv = \frac{1}{2}\iiint \boldsymbol{D}\cdot\boldsymbol{E}\, dv$

The *flux linkage* of an inductor defines the total magnetic flux that links the current. If the magnetic flux produced by a given current links that same current, the resulting inductance is defined as a *self inductance*. If the magnetic flux produced by a given current links the current in another circuit, the resulting inductance is defined as a *mutual inductance*.

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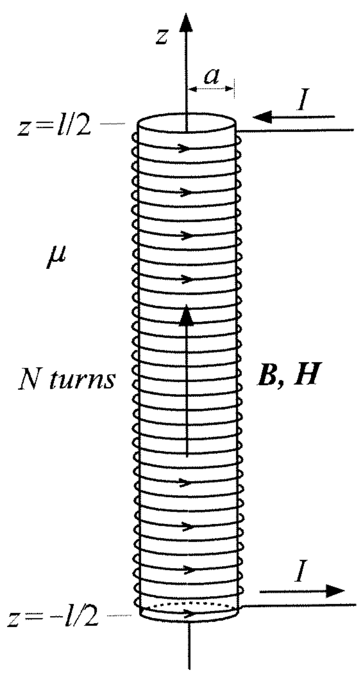
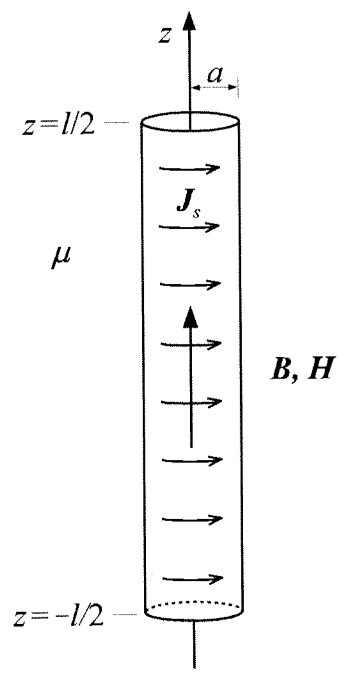
Solenoid

A *solenoid* is a cylindrically shaped current carrying coil. The solenoid is the magnetic field equivalent to the parallel plate capacitor for electric fields. Just as the parallel plate capacitor concentrates the electric field between the plates, the solenoid concentrates the magnetic field within the coil. For the purpose of determining the solenoid magnetic field, the solenoid of length l and radius a which is tightly wound with N turns can be modeled as an equivalent uniform surface current on the cylinder surface.



Solenoid
length = l, radius = a
number of turns = N
current = I



Equivalent uniform
surface current model
$J_o = NI/l$

The equivalent uniform surface current density (J_o) for the solenoid is found by spreading the total current of NI over the length l.

$$\boldsymbol{J}_s = J_o \hat{\boldsymbol{\phi}} = \frac{NI}{l} \hat{\boldsymbol{\phi}}$$

The Biot-Savart law integral to determine the magnetic field of the solenoid equivalent surface current is

$$\boldsymbol{H} = \frac{1}{4\pi} \iint_S \frac{\boldsymbol{J}_s \times (\boldsymbol{R} - \boldsymbol{R}')}{R^3} \, ds'$$

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To determine the characteristics of the magnetic field inside the solenoid, we choose the field point P on the solenoid axis (z-axis).

$$\boldsymbol{J}_s = \frac{NI}{l}\hat{\boldsymbol{\phi}}$$

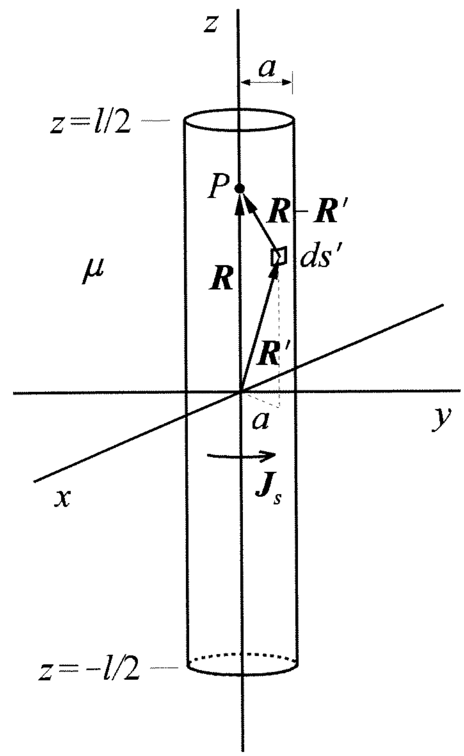
$$ds' = a\,d\phi'\,dz'$$

$$\boldsymbol{R} = z\hat{\boldsymbol{z}}$$

$$\boldsymbol{R}' = a\hat{\boldsymbol{r}} + z'\hat{\boldsymbol{z}}$$

$$\boldsymbol{R} - \boldsymbol{R}' = -a\hat{\boldsymbol{r}} + (z - z')\hat{\boldsymbol{z}}$$

$$R_o = |\boldsymbol{R} - \boldsymbol{R}'| = \sqrt{a^2 + (z - z')^2}$$



The cross product in the numerator of the Biot-Savart law integral is

$$\boldsymbol{J}_s \times (\boldsymbol{R} - \boldsymbol{R}') = \left(\frac{NI}{l}\hat{\boldsymbol{\phi}}\right) \times \left[-a\hat{\boldsymbol{r}} + (z - z')\hat{\boldsymbol{z}}\right]$$

$$= \frac{NI}{l}\left[a\hat{\boldsymbol{z}} + (z - z')\hat{\boldsymbol{r}}\right]$$

The $\hat{\boldsymbol{r}}$ unit vector written in terms of rectangular coordinate unit vectors is

$$\hat{\boldsymbol{r}} = \cos\phi'\hat{\boldsymbol{x}} + \sin\phi'\hat{\boldsymbol{y}}$$

The Biot-Savart law integral for the solenoid becomes

$$\boldsymbol{H} = \frac{1}{4\pi}\int_{-l/2}^{l/2}\int_{0}^{2\pi} \frac{\frac{NI}{l}\left[(z - z')\overset{0}{\cos\phi'}\hat{\boldsymbol{x}} + (z - z')\overset{0}{\sin\phi'}\hat{\boldsymbol{y}} + a\hat{\boldsymbol{z}}\right]}{\left[a^2 + (z - z')^2\right]^{3/2}}\, a\,d\phi'\,dz'$$

$$= \frac{NIa^2}{4\pi l}\hat{\boldsymbol{z}}\int_{-l/2}^{l/2}\int_{0}^{2\pi}\frac{d\phi'\,dz'}{\left[a^2 + (z - z')^2\right]^{3/2}}$$

$$= \frac{NIa^2}{4\pi l}\hat{\boldsymbol{z}}(2\pi)\int_{-l/2}^{l/2}\frac{dz'}{\left[a^2 + (z - z')^2\right]^{3/2}}$$

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The remaining z' integration is the same form as that found for the current segment on the z-axis. The result of the integration is

$$\boldsymbol{H} = \frac{NIa^2}{2l}\hat{z}\frac{1}{a^2}\left[\frac{z+l/2}{\sqrt{a^2+(z+l/2)^2}} - \frac{z-l/2}{\sqrt{a^2+(z-l/2)^2}}\right]$$

$$= \frac{NI}{2l}\left[\frac{z+l/2}{\sqrt{a^2+(z+l/2)^2}} - \frac{z-l/2}{\sqrt{a^2+(z-l/2)^2}}\right]\hat{z}$$

The magnetic field at the center of the solenoid ($z = 0$) is

$$\boldsymbol{H} = \frac{NI}{2l}\left[\frac{l}{\sqrt{a^2+(l/2)^2}}\right]\hat{z} = \frac{NI}{2\sqrt{a^2+(l/2)^2}}\hat{z}$$

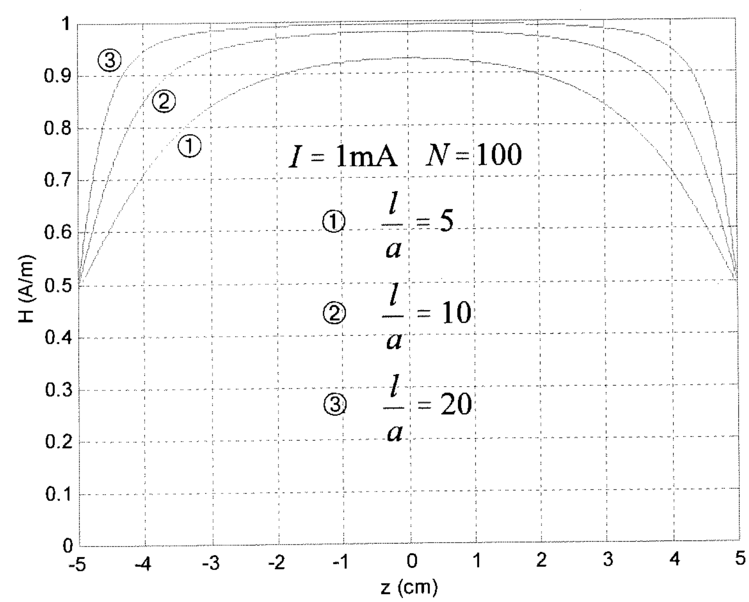
At either end of the solenoid ($z = +l/2, -l/2$), the magnetic field is

$$\boldsymbol{H} = \frac{NI}{2l}\left[\frac{l}{\sqrt{a^2+l^2}}\right]\hat{z} = \left[\frac{NI}{2\sqrt{a^2+l^2}}\right]\hat{z}$$

For a long solenoid ($l \gg a$), the approximate magnetic field values at the center and at the ends of the solenoid are

$$\boldsymbol{H} \approx \frac{NI}{l}\boldsymbol{a}_z \qquad \text{(center)} \qquad\qquad \boldsymbol{H} \approx \frac{NI}{2l}\boldsymbol{a}_z \qquad \text{(at either end)}$$

Thus, the magnetic field at the ends of a long solenoid is approximately half that at the center of the solenoid. However, the magnetic field over the length is a long solenoid is relatively constant. At the ends of the long solenoid, the magnetic field falls rapidly to about one-half of the peak value.



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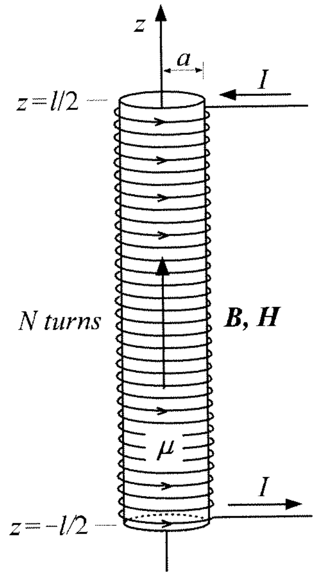
Example (Self inductance / long solenoid)

Given the long solenoid ($l \gg a$), the magnetic field throughout the solenoid can be assumed to be constant.

$$H = \frac{NI}{l} \qquad B = \mu H = \mu \frac{NI}{l}$$



According to the definition of inductance, the inductance of the long solenoid is

$$L = \frac{\Lambda}{I} = \frac{N\psi_m}{I}$$

The total magnetic flux through the solenoid is

$$\psi_m = \iint \mathbf{B} \cdot d\mathbf{s} = BA = \mu \frac{NI}{l} \pi a^2$$

Inserting the total magnetic flux expression into the inductance equation yields

$$L = \frac{\mu N^2 I \pi a^2}{Il} = \frac{\mu N^2 \pi a^2}{l} \quad \text{(H)} \qquad \text{(long solenoid)}$$

Note that the inductance of the long solenoid is directly proportional to the permeability of the medium inside the *core* of the solenoid. By using a ferromagnetic material such as iron as the solenoid core, the inductance can be increased significantly given the large relative permeability of a ferromagnetic material.

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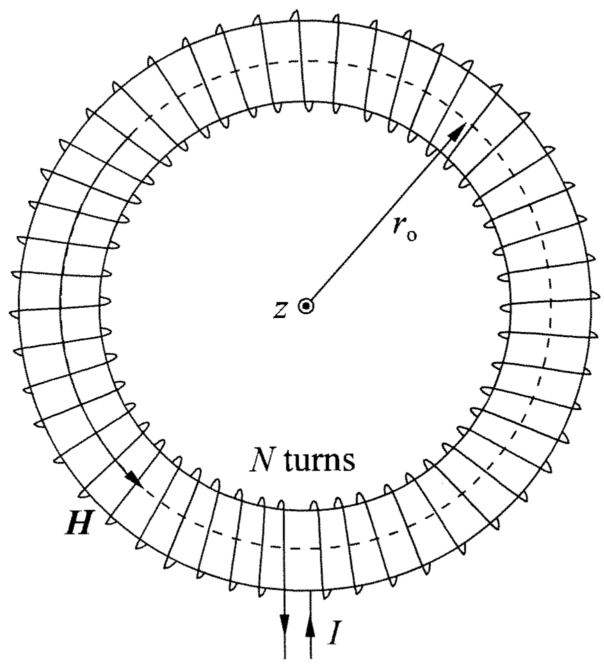
Toroid

Another commonly encountered magnetic energy storage geometry is the *toroid*. A toroid is formed by wrapping a conductor around a ring of uniform cross-section (typically circular cross-section). The distance from the center of the ring to the center of the ring cross-section is defined as the *mean radius* r_o. Given a circular cross-section of radius a, if the mean radius is large relative to the radius of the cross section ($r_o \gg a$), then the toroid may be viewed as a long solenoid bent into the shape of a circle (the magnetic field within the toroid may be assumed to be uniform). Application of Ampere's law on the mean radius path gives



$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = \oint_L H_\phi \, dl = H_\phi \oint_L dl = H_\phi (2\pi r_o) = I_{enclosed} = NI$$

Solving for the toroid magnetic field yields

$$H_\phi = \frac{NI}{2\pi r_o} = \frac{NI}{l}$$

where $l = 2\pi r_o$ is the equivalent length of the toroid. The magnetic field at any point within the toroid is the same as that found at the center of the long solenoid. Thus, the self-inductance of the toroid is the same as the equivalent long solenoid (replace l with $2\pi r_o$).

$$L = \frac{\mu N^2 \pi a^2}{2\pi r_o} = \frac{\mu N^2 a^2}{2 r_o} \qquad \text{(toroid)}$$

The primary advantage of the toroid over the solenoid is the confinement of the magnetic field within the toroid as opposed to the solenoid which produces magnetic fields external to the coil. Also, the toroid does not suffer from the end effects (fringing) seen in the solenoid.

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Coaxial Transmission Line

The magnetic field between the conductors of coaxial transmission line was shown using Ampere's law to be

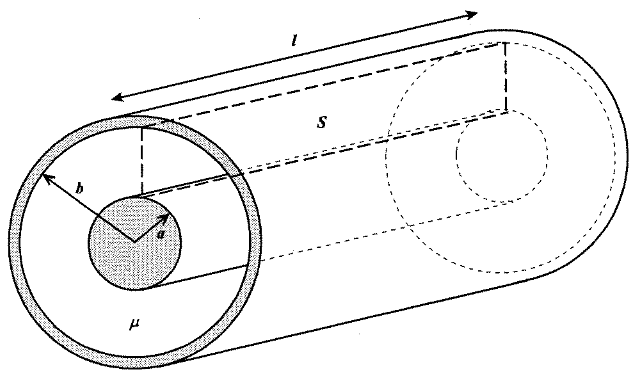
$$\boldsymbol{H} = \frac{I}{2\pi r}\hat{\phi} \qquad (a<r<b)$$

where I is the total current in each conductor (flowing in opposite directions in the two conductors). The self inductance of the coaxial transmission line is found by determining the total magnetic flux that links the transmission line current (total flux linkage). The flux linkage for the coaxial transmission line is found by integrating the magnetic flux density between the conductors over the surface S shown in the figure below.

$$\Lambda = \psi_m = \iint_S \boldsymbol{B}\cdot d\boldsymbol{s}$$

$$\boldsymbol{B} = \frac{\mu I}{2\pi r}\hat{\phi}$$

$$d\boldsymbol{s} = dr\,dz\,\hat{\phi}$$



$$\Lambda = \int_0^l\int_a^b \frac{\mu I}{2\pi r}\hat{\phi}\cdot dr\,dz\,\hat{\phi} = \frac{\mu I}{2\pi}\int_a^b \frac{dr}{r}\int_0^l dz = \frac{\mu I l}{2\pi}\ln\left(\frac{b}{a}\right)$$

The inductance of a length l of coaxial transmission line is

$$L = \frac{\Lambda}{I} = \frac{\mu l}{2\pi}\ln\left(\frac{b}{a}\right) \quad \text{(H)}$$

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Time varying Fields and Maxwell's Equation

The general form of the four basic laws governing electromagnetic fields (Maxwell's equations) are shown below. All of the vector field, flux, current and charge terms in Maxwell's equations are, in general, functions of both time and space [e.g., $\boldsymbol{E}(x,y,z,t)$]. The form of these quantities is referred to as the *instantaneous form* (we can describe the fields at any point in time and space). The instantaneous form of Maxwell's equations may be used to analyze electromagnetic fields with any arbitrary time-variation.

Maxwell's Equations [instantaneous, differential form]

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{(Faraday's law)}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad \text{(Ampere's law)}$$

$$\nabla \cdot \boldsymbol{D} = \rho_v \qquad \text{(Gauss's law)}$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{(Gauss's law for magnetic fields)}$$

Maxwell's Equations [instantaneous, integral form]

$$\oint_L \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{\partial}{\partial t} \iint_S \boldsymbol{B} \cdot d\boldsymbol{s} \qquad \text{(Faraday's law)}$$

$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = \iint_S \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot d\boldsymbol{s} \qquad \text{(Ampere's law)}$$

$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = \oint_V \rho_v \, dv = Q \qquad \text{(Gauss's law)}$$

$$\oint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0 \qquad \text{(Gauss's law for magnetic fields)}$$

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Note that the dynamic form of Maxwell's equations contain time-derivatives in Faraday's law and Ampere's law. We will focus on these time-derivatives and discuss how these terms come about in dynamic (electromagnetic) fields.

Complete Form of Faraday's Law (Dynamic Fields)

The complete form of Faraday's law, valid for both static and dynamic fields, is defined in terms of a quantity known as the *electromotive force* (emf) which has units of volts. In an electric circuit, the emf is the force which sets the charge in motion (forcing function for the current, voltage source). The emf in Faraday's law an induced forcing function generated by time-varying fields. In general, the integral of the electric field around a closed circuit yields the total emf (V_{emf}) in the integration path.

$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l}$$

The dynamic form of Faraday's law can be defined in terms of electromotive force as:

<u>Faraday's Law</u> – a time-changing magnetic flux through a closed circuit induces an emf in the circuit (closed circuit – induced current, open circuit – induced voltage).

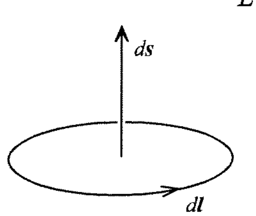
$$V_{emf} = -\frac{d\psi_m}{dt}$$

The emf is an equal and opposite reaction to the flux change (Lenz's law)

$$\psi_m = \iint_S \mathbf{B} \cdot d\mathbf{s} \qquad V_{emf} = -\frac{d}{dt}\iint_S \mathbf{B} \cdot d\mathbf{s}$$

$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\iint_S \mathbf{B} \cdot d\mathbf{s}$$

Faraday's law (integral form)



The unit normal associated with the differential surface *ds* is related to the unit vector of the differential length *dl* by the right hand rule.

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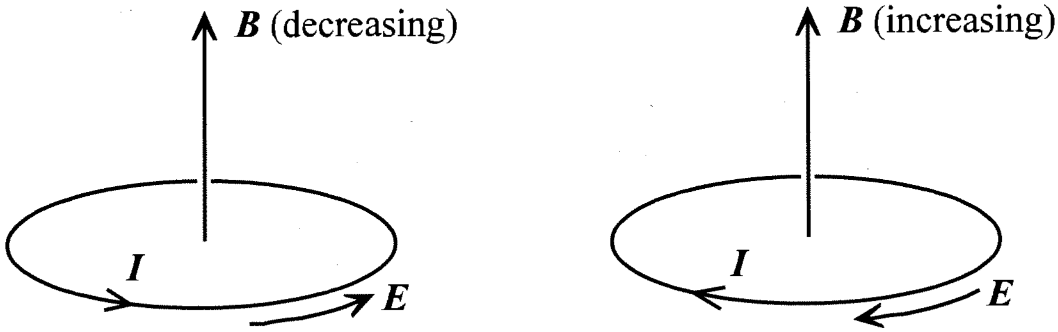
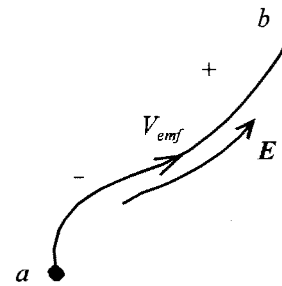
Example (Faraday's law induction, wire loop in a time changing ***B***)

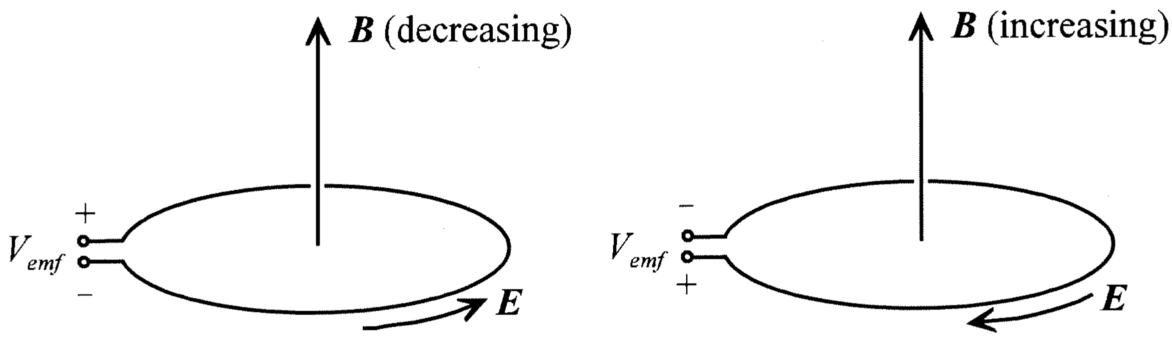
For the closed loop, the flux produced by the induced current opposes the change in ***B***.



For the open-circuited loop, the polarity of the induced emf is defined by the emf line integral.

$$V_{emf} = \int_a^b \mathbf{E} \cdot d\mathbf{l}$$





Induction Types

1. Stationary circuit / time-varying ***B*** (*transformer induction*).
2. Moving circuit / static ***B*** (*motional induction*).
3. Moving circuit / time-varying ***B***
(*general case, transformer and motional induction*).

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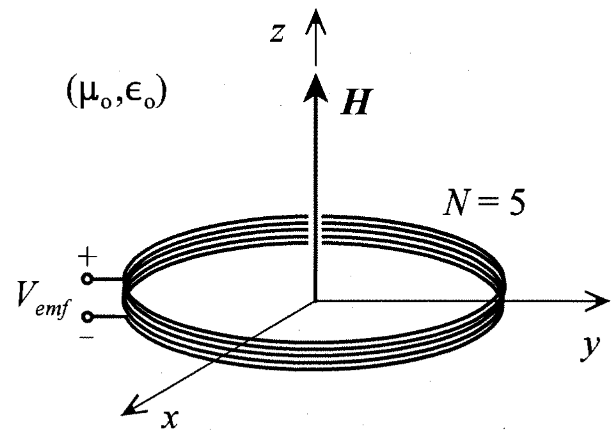
Example (transformer induction - AM loop antenna)

A 5-turn circular wire loop ($N = 5$) of radius $a = 0.4$m lies centered in the x-y plane with its axis along the z-axis. The loop is located in a time-varying vector magnetic field defined by $\boldsymbol{H} = H_o \cos(\omega t)\hat{z}$ where $H_o = 200$ μA/m and $f = 1$ MHz. Determine the emf induced at the loop terminals.

$$V_{emf} = -N \frac{d}{dt} \iint_S \boldsymbol{B} \cdot d\boldsymbol{s}$$



Note that we have modified Faraday's law to account for the multiple turns in the loop (by multiplying the single-turn formula by N). Since the loop is stationary, $d\boldsymbol{s}$ is not time-dependent so that the derivative with respect to time can be brought inside the integral.

$$V_{emf} = -N \iint_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s} \qquad \text{(transformer induction)}$$

The time derivative is written as a partial derivative since the magnetic flux density is, in general, a function of both time and space. The polarity of the induced emf is assigned when the direction of $d\boldsymbol{s}$ is chosen. If we choose $d\boldsymbol{s} = \hat{z}\,ds$ (then $d\boldsymbol{l} = \hat{\phi}$ dl for the line integral of $\boldsymbol{E}$), the polarity of the induced emf is that shown above. For this problem, both $\boldsymbol{B}$ [$\boldsymbol{B} = \mu_o \boldsymbol{H}$] and $d\boldsymbol{s}$ are $\hat{z}$-directed so that the dot product in the transformer induction integral is one.

$$V_{emf} = -N \iint_S \frac{\partial B}{\partial t}\, ds = -N\mu_o \iint_S \frac{\partial H}{\partial t}\, ds$$

$$\frac{\partial H}{\partial t} = H_o \frac{\partial}{\partial t}(\cos\omega t) = H_o(-\omega \sin\omega t)$$

$$= -\omega H_o \sin\omega t \qquad \left(\begin{array}{c}\text{function of time,}\\ \text{not position}\end{array}\right)$$

Since the partial derivative of H with respect to time is independent of position, it can be brought outside the integral. The resulting integral of ds over the surface S yields the area of the loop so that

$$V_{emf} = -N\mu_o \frac{\partial H}{\partial t} \iint_S ds = -N\mu_o \frac{\partial H}{\partial t} A$$

where A is the area of the loop ($A = \pi a^2$).

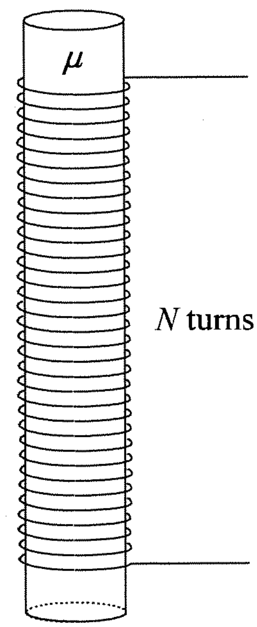
$$V_{emf} = -N\mu_o(-\omega H_o \sin\omega t)\pi a^2$$

$$= \omega\mu_o N\pi a^2 H_o \sin\omega t \qquad \text{(oscillating emf)}$$

$$|V_{emf}| = \omega\mu_o N\pi a^2 H_o$$

$$= (2\pi\, 10^6)(4\pi\times 10^{-7})5\pi(0.4)^2(200\times 10^{-6})$$

$$= 3.97 \text{ mV}$$



Note that the emf induced at the terminals of the antenna is a scaled, phase-shifted version of the magnetic field. A typical AM antenna achieves a larger induced emf by employing a large number of wire turns around a ferrite core.

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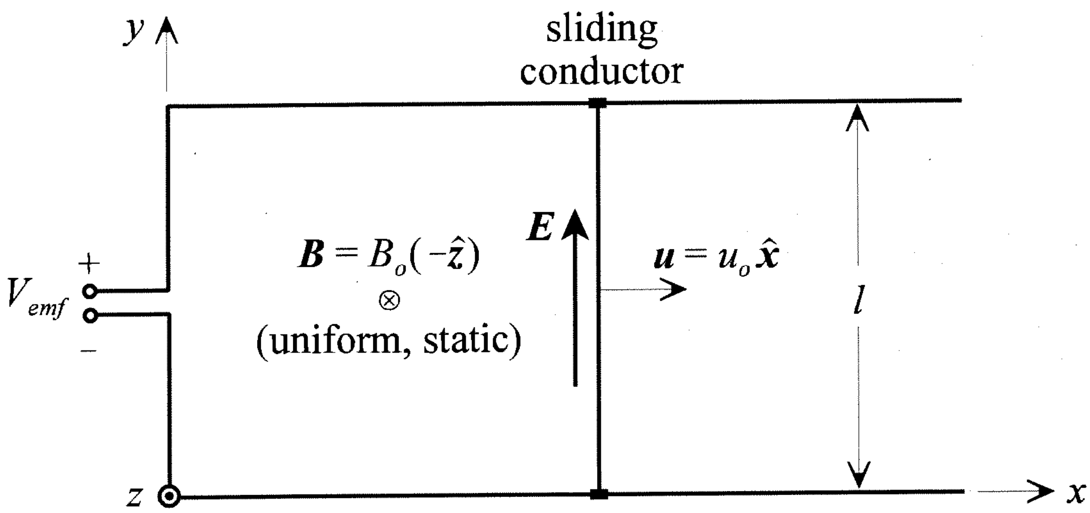
Example (motional induction – moving conductor / static $\boldsymbol{B}$)



A particle of charge q moving with velocity $\boldsymbol{u}$ in a uniform $\boldsymbol{B}$ experiences a force given by

$$\boldsymbol{F} = q(\boldsymbol{u} \times \boldsymbol{B}) \qquad \text{(Lorentz force equation)}$$

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{q} = \boldsymbol{u} \times \boldsymbol{B} \qquad \begin{pmatrix} \text{emf electric field} \\ \text{motional induction} \end{pmatrix}$$

$$= [u_o\hat{\boldsymbol{x}}] \times [B_o(-\hat{\boldsymbol{z}})] = u_oB_o\hat{\boldsymbol{y}}$$

From Faraday's law,

$$V_{emf} = \oint \boldsymbol{E} \cdot d\boldsymbol{l} = \oint (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} \qquad \begin{pmatrix} \text{motional induction} \\ \text{"flux cutting emf"} \end{pmatrix}$$

Choosing $d\boldsymbol{l}$ counterclockwise assigns the induced emf polarity as shown above. On the moving conductor, the differential length is $d\boldsymbol{l} = dy\,\hat{\boldsymbol{y}}$.

$$V_{emf} = \int_0^l (u_oB_o\hat{\boldsymbol{y}}) \cdot (dy\,\hat{\boldsymbol{y}}) = u_oB_o \int_0^l dy = u_oB_ol$$

Note that a uniform velocity yields a DC voltage. An oscillatory motion (back and forth) could be used to produce a sinusoidal voltage.

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Example (General induction – moving conductor / time-varying $\boldsymbol{B}$)

Using the same geometry as the last example, assume that the magnetic flux density is $\boldsymbol{B} = B_o \cos \omega t\,(-\hat{\boldsymbol{z}})$.

$$V_{emf} = -\iint_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s} + \oint_L (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} \qquad \text{(general induction)}$$

Choose $d\boldsymbol{l}$ counterclockwise $\Rightarrow$ $d\boldsymbol{s}$ out

$d\boldsymbol{l} = dy\,\hat{\boldsymbol{y}}$ (on moving conductor) $\quad d\boldsymbol{s} = dxdy\,\hat{\boldsymbol{z}}$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\omega B_o \sin \omega t\,(-\hat{\boldsymbol{z}}) = \omega B_o \sin \omega t\,\hat{\boldsymbol{z}}$$

$$\boldsymbol{u} \times \boldsymbol{B} = [u_o \hat{\boldsymbol{x}}] \times [B_o \cos \omega t\,(-\hat{\boldsymbol{z}})] = u_o B_o \cos \omega t\,\hat{\boldsymbol{y}}$$

$$V_{emf} = -\int_0^x \int_0^l (\omega B_o \sin \omega t\,\hat{\boldsymbol{z}}) \cdot (dx\,dy\,\hat{\boldsymbol{z}}) + \int_0^l (u_o B_o \cos \omega t\,\hat{\boldsymbol{y}}) \cdot (dy\,\hat{\boldsymbol{y}})$$

$$= -\omega B_o \sin \omega t \int_0^x \int_0^l dx\,dy + u_o B_o \cos \omega t \int_0^l dy$$

$$= -\omega x l B_o \sin \omega t + u_o l B_o \cos \omega t$$

If we let $x = 0$ at $t = 0$ be our reference, then $x = u_o t$ and

$$V_{emf} = u_o l B_o\,[\cos \omega t - \omega t \sin \omega t]$$

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Summary of Induction Formulas

$$V_{emf} = \oint_L \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt} \iint_S \boldsymbol{B} \cdot d\boldsymbol{s} \qquad \text{Faraday's law (integral form)}$$

$$V_{emf} = V_t = \oint_L \boldsymbol{E}_t \cdot d\boldsymbol{l} = -\iint_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s} \qquad \left(\begin{array}{c}\text{transformer} \\ \text{induction only}\end{array}\right)$$

$$V_{emf} = V_m = \oint_L \boldsymbol{E}_m \cdot d\boldsymbol{l} = \oint_L (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} \qquad \left(\begin{array}{c}\text{motional} \\ \text{induction only}\end{array}\right)$$

$$V_{emf} = V_t + V_m = -\iint_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s} + \oint_L (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} \qquad \left(\begin{array}{c}\text{general} \\ \text{case}\end{array}\right)$$

$$= \oint_L (\boldsymbol{E}_t + \boldsymbol{E}_m) \cdot d\boldsymbol{l} = \oint_L \boldsymbol{E} \cdot d\boldsymbol{l}$$

$$\boldsymbol{E} = \boldsymbol{E}_t + \boldsymbol{E}_m$$

$\boldsymbol{E}_t$ = transformer emf electric field

$\boldsymbol{E}_m$ = motional emf electric field

$\boldsymbol{E}$ = total emf electric field

Note that Faraday's law can be written in a variety of forms. The general formula (top formula) is always valid but we see that the line and surface integrals may each implicitly contain separate transformer and motional emf contributions.

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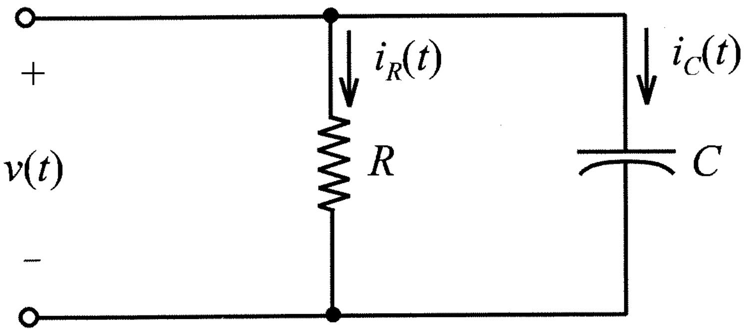
Displacement Current
(Maxwell's contribution to Maxwell's equations)

The concept of displacement current can be illustrated by considering the currents in a simple parallel RC network (assume ideal circuit elements, for simplicity).

$i_R(t)$ – conduction current

$i_C(t)$ – displacement current



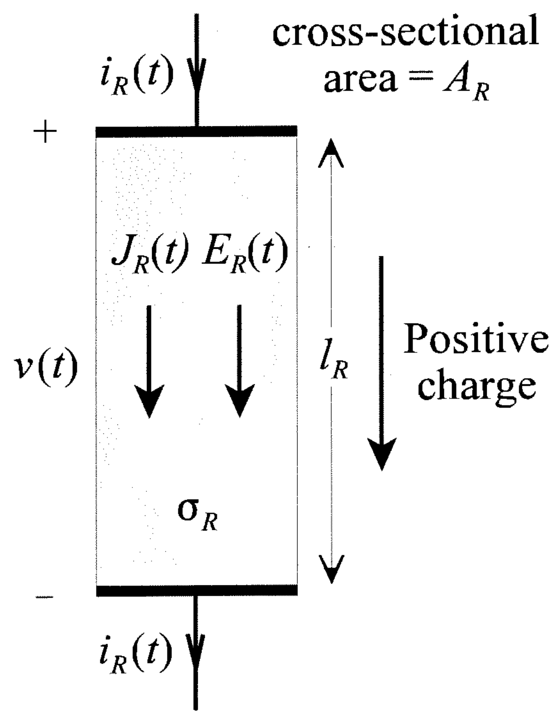
From circuit theory $\Rightarrow$ $i_R(t) = \dfrac{v(t)}{R}$ $\qquad i_C(t) = C\dfrac{dv(t)}{dt}$

In the resistor, the conduction current model is valid ($\boldsymbol{J}_R = \sigma_R \boldsymbol{E}_R$). The ideal resistor electric field ($\boldsymbol{E}_R$) and current density ($\boldsymbol{J}_R$) are assumed to be uniform throughout the volume of the resistor.

$$J_R(t) = \frac{i_R(t)}{A_R} = \sigma_R E_R(t) = \sigma_R \frac{v(t)}{l_R}$$

$$i_R(t) = \frac{\sigma_R A_R}{l_R} v(t) = \frac{v(t)}{R}$$

$$R = \frac{l_R}{\sigma_R A_R}$$



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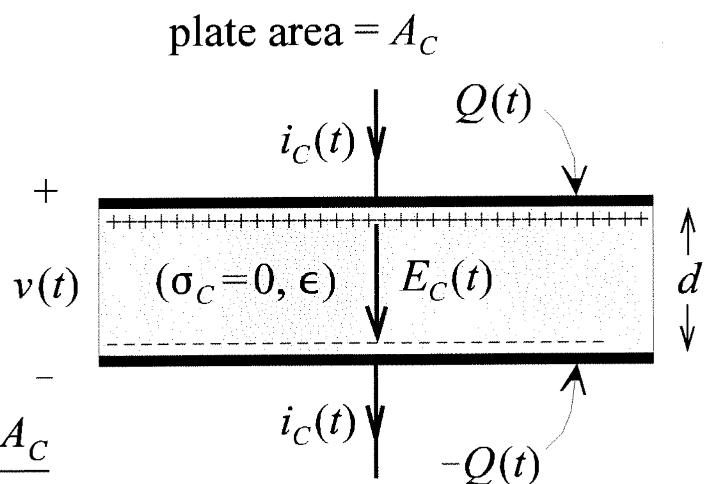
The conduction current model does not characterize the capacitor current. The ideal capacitor is characterized by large, closely-spaced plates separated by a perfect insulator ($\sigma_C = 0$) so that no charge actually passes throught the dielectric [$\boldsymbol{J}_C(t) = \sigma_C \boldsymbol{E}_C(t)$]. The capacitor current measured in the connecting wires of the capacitor is caused by the charging and discharging the capacitor plates. Let $Q(t)$ be the total capacitor charge on the positive plate.

$$i_C(t) = \frac{dQ(t)}{dt}$$

$$Q(t) = Cv(t)$$

$$i_C(t) = C\frac{dv(t)}{dt}$$



$$v(t) = E_C(t)\,d \qquad C = \frac{\epsilon A_C}{d}$$

$$i_C(t) = \frac{\epsilon A_C}{d}\frac{d}{dt}[E_C(t)d] = \epsilon A_C\frac{dE_C(t)}{dt} = A_C\frac{dD_C(t)}{dt}$$

$$J_C(t) = \frac{i_C(t)}{A_C} = \frac{dD_C(t)}{dt} \qquad \left(\begin{array}{c}\text{displacement}\\ \text{current density}\end{array}\right)$$

Based on these results, the static version of Ampere's law must be modified for dynamic fields to include conduction current AND displacement current. Note that displacement current does not exist under static conditions. The general form for current density in the dynamic field problem is

$$\boldsymbol{J}_{total} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad \text{where} \qquad \boldsymbol{J} = \sigma\boldsymbol{E} + \rho_v \boldsymbol{u}$$

$\frac{\partial \boldsymbol{D}}{\partial t}$: displacement current; $\sigma\boldsymbol{E}$: conduction current + $\rho_v\boldsymbol{u}$: convection current

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Complete Form of Ampere's Law (Dynamic Fields)

Given the definition of displacement current, the complete form of Ampere's law for dynamic fields can be written.

$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = I_{enc} = \iint_S \boldsymbol{J}_{total} \cdot d\boldsymbol{s} = \iint_S \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot d\boldsymbol{s}$$

$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = \iint_S \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot d\boldsymbol{s}$$

Ampere's law
(integral form)

The corresponding differential form of Ampere's law is found using Stoke's theorem.

$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = \iint_S (\nabla \times \boldsymbol{H}) \cdot d\boldsymbol{s} = \iint_S \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot d\boldsymbol{s}$$

Since the two surface integrals above are valid for any surface *S*, we may equate the integrands.

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Ampere's law
(differential form)

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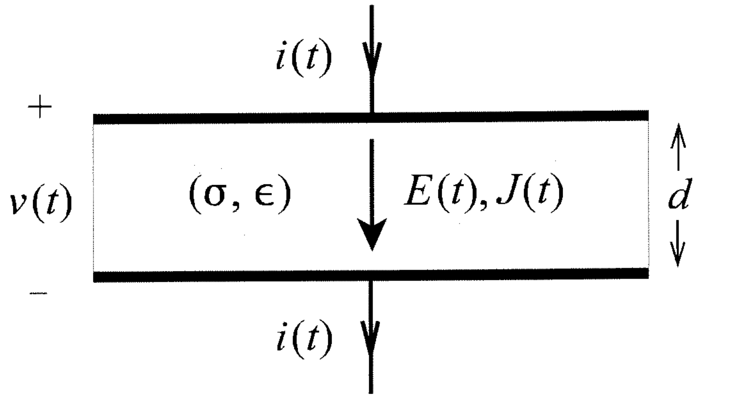
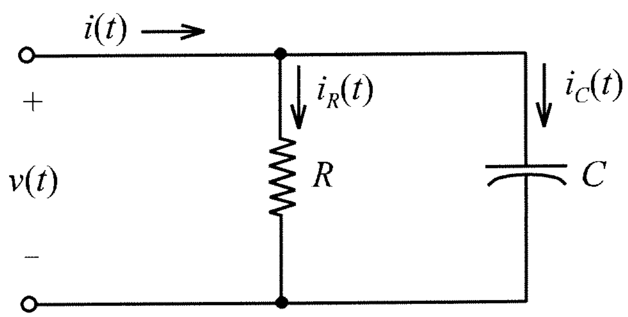
Example (Ampere's law, non-ideal capacitor)

The previously considered parallel RC network represents the equivalent circuit of a parallel plate capacitor with an imperfect insulating material between the capacitor plates (finite conductivity).



Capacitor with imperfect insulating material (assume E, J are uniform)



C – models charge storage (displacement current)

R – models leakage current (conduction current)

Equivalent circuit

Let the applied voltage be a sinusoid. $\Rightarrow$ $V(t) = V_o \sin\omega t$

The resulting electric field in the capacitor is given by

$$E(t) = \frac{v(t)}{d} = \frac{V_o}{d}\sin\omega t = E_o \sin\omega t \qquad E_o = \frac{V_o}{d} \quad \left(\begin{array}{c}\text{peak capacitor}\\ \text{electric field}\end{array}\right)$$

The conduction current in the non-ideal capacitor is given by the product of the insulator conductivity and the capacitor electric field.

$$J(t) = \sigma E(t) = \sigma \frac{V_o}{d}\sin\omega t = \sigma E_o \sin\omega t = J_o \sin\omega t$$

$$J_o = \frac{\sigma V_o}{d} = \sigma E_o \qquad \left(\begin{array}{c}\text{peak capacitor conduction}\\ \text{current density}\end{array}\right)$$

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The displacement current in the non-ideal capacitor is given by the partial derivative of the capacitor flux density.

$$\frac{\partial D(t)}{\partial t} = \epsilon \frac{\partial E(t)}{\partial t} = \epsilon \frac{\partial}{\partial t}\left(\frac{V_o}{d} \sin\omega t \right) = \omega \epsilon \frac{V_o}{d} \cos\omega t = D_o \cos\omega t$$

$$D_o = \omega \epsilon \frac{V_o}{d} = \omega \epsilon E_o \qquad \left(\begin{matrix} \text{peak capacitor displacement} \\ \text{current density} \end{matrix} \right)$$

Note that:

1. The peak conduction current density is independent of frequency.
2. The peak displacement current density is directly proportional to frequency.
3. The displacement current density leads the conduction current density by 90°.

Conduction currents and displacement currents in problems involving time-harmonic electromagnetic fields exhibit the same behavior seen in the non-ideal parallel plate capacitor. Namely,

$$|\boldsymbol{J}| = \sigma E \qquad \left| \frac{\partial D(t)}{\partial t} \right| = \omega \epsilon E$$

Thus, the relative magnitudes of the two terms σ and $\omega\epsilon$ dictate if one type of current dominates the other. Since typical material permittivities are in the 1-100 pF/m range, the displacement current density is typically negligible at low frequencies in comparison to the conduction current density (especially in good conductors). At high frequencies, the displacement current density becomes more significant and will typically dominate the conduction current density in good insulators. The classification of a material as a "good conductor" or a "good insulator" can be made based on the relative size of σ and $\omega\epsilon$.

$$\sigma \gg \omega \epsilon \qquad \text{good conductor} \qquad \left(\begin{matrix} \text{displacement current} \\ \text{is negligible} \end{matrix} \right)$$

$$\sigma \ll \omega \epsilon \qquad \text{good insulator} \qquad \left(\begin{matrix} \text{conduction current} \\ \text{is negligible} \end{matrix} \right)$$