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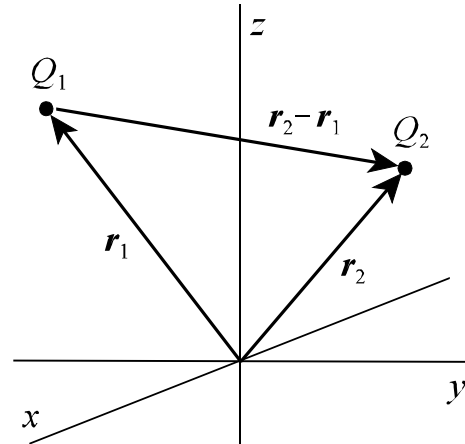**www.ahmed.ucoz.org**

Electrostatic Fields

Electrostatic fields are static (time-invariant) electric fields produced by static (stationary) charge distributions. The mathematical definition of the electrostatic field is derived from *Coulomb's law* which defines the vector force between two point charges.

<u>Coulomb's Law</u>

$$F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_o R^2} a_{R_{12}}$$

Q_1, Q_2	point charges (C)
F_{12}	vector force (N) on Q_2 due to Q_1
r_1, r_2	vectors locating Q_1 and Q_2
$R_{12} = r_2 - r_1$	vector pointing from Q_1 to Q_2
$R = |R_{12}| = |r_2 - r_1|$	separation distance (m) between Q_1 and Q_2
$a_{R_{12}} = |R_{12}|/R$	unit vector pointing from Q_1 to Q_2
ϵ_o	free space (vacuum) permittivity [8.854×10^{-12} F/m]

Coulomb's law can also be written as

$$F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_o R^3} R_{12} = \frac{Q_1 Q_2 (r_2 - r_1)}{4\pi\epsilon_o |r_2 - r_1|^3}$$

Note that the unit vector direction is defined according to which charge is exerting the force and which charge is experiencing the force. This convention assures that the resulting vector force always points in the appropriate direction (opposite charges attract, like charges repel).
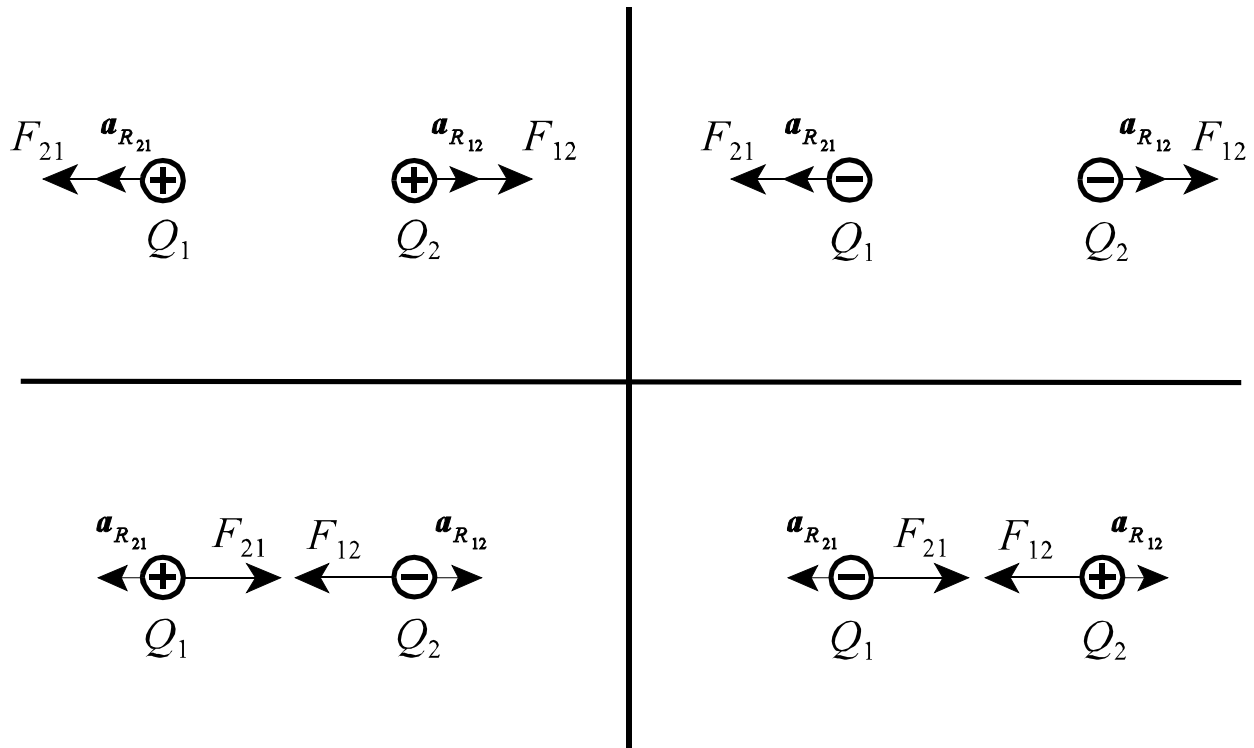

The point charge is a mathematical approximation to a very small volume charge. The definition of a point charge assumes a finite charge located at a point (zero volume). The point charge model is applicable to small charged particles or when two charged bodies are separated by such a large distance that these bodies appear as point charges to each other.

Given multiple point charges in a region, the *principle of superposition* is applied to determine the overall vector force on a particular charge. The total vector force acting on the charge equals the vector sum of the individual forces.

Force Due to Multiple Point Charges

Given a point charge Q in the vicinity of a set of N point charges (Q_1, Q_2,..., Q_N), the total vector force on Q is the vector sum of the individual forces due to the N point charges.

$F \equiv$ total vector force on Q
  due to Q_1, Q_2,..., Q_N
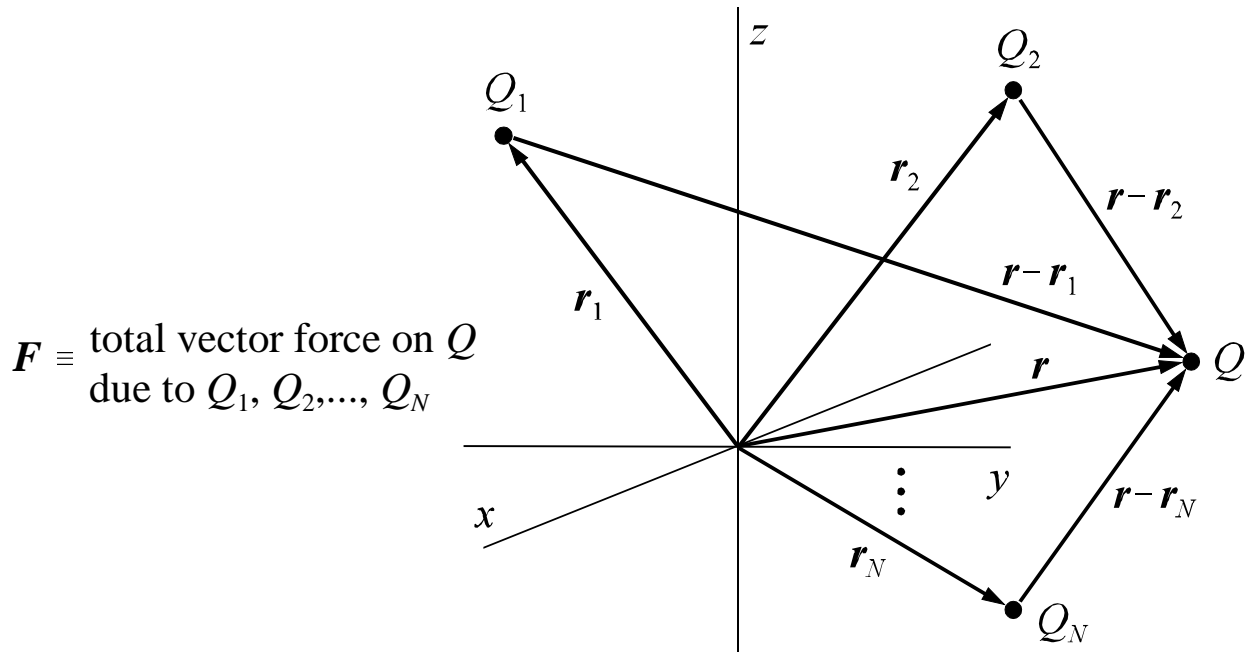
$$F = \frac{QQ_1(r - r_1)}{4\pi\epsilon_o |r - r_1|^3} + \frac{QQ_2(r - r_2)}{4\pi\epsilon_o |r - r_2|^3} + ... + \frac{QQ_N(r - r_N)}{4\pi\epsilon_o |r - r_N|^3}$$

$$F = \frac{Q}{4\pi\epsilon_o} \sum_{k=1}^{N} Q_k \frac{(r - r_k)}{|r - r_k|^3}$$

Electric Field

According to Coulomb's law, the vector force between two point charges is directly proportional to the product of the two charges. Alternatively, we may view each point charge as producing a force field around it (*electric field*) which is proportional to the point charge magnitude. When a ***positive*** *test charge Q* is placed at the point *P* (the *field point*) in the force field of a point charge Q' located at the point P', the force per unit charge experienced by the test charge *Q* is defined as the electric field at the point *P*. Given our convention of using a positive test charge, **the direction of the vector electric field is the direction of the force on positive charge**. A convention has been chosen where the *source coordinates* (location of the source charge) are defined by primed coordinates while the *field coordinates* (location of the field point) are defined by unprimed coordinates.

Q' - point charge producing
      the electric field

Q - positive test charge used
      to measure the electric field

r' - locates the *source point*
      (location of source charge Q')

r - locates the *field point*
      (location of test charge Q)
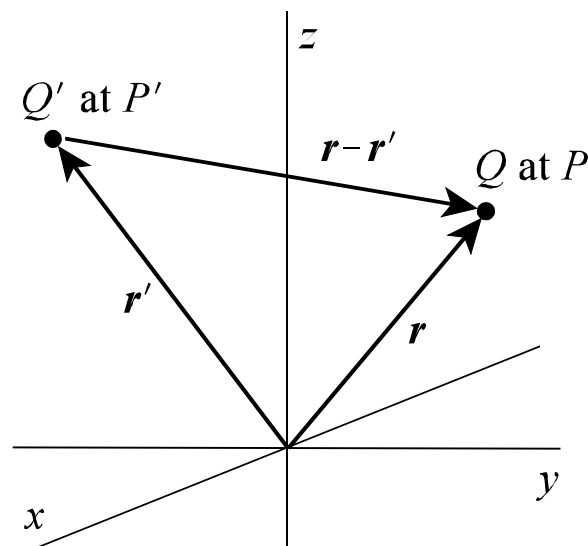
From Coulomb's law, the force on the test charge Q at r due to the charge Q' at r' is

$$F(r) = \frac{QQ'(r - r')}{4\pi\epsilon_o |r - r'|^3} \quad (N)$$

The vector electric field intensity E at r (force per unit charge) is found by dividing the Coulomb force equation by the test charge Q.

$$E(r) \equiv \frac{F(r)}{Q} = \frac{Q'(r - r')}{4\pi\epsilon_o |r - r'|^3} \quad (\text{N/C})$$

Note that the electric field produced by Q' is independent of the magnitude of the test charge Q. The electric field units [Newtons per Coulomb (N/C)] are normally expressed as Volts per meter (V/m) according to the following equivalent relationship:

$$\frac{N}{C} = \frac{J/m}{C} = \frac{J/C}{m} = \frac{V}{m}$$

For the special case of a point charge at the origin ($r' = 0$), the electric field reduces to the following spherical coordinate expression:

$$E(r) = \frac{Q'r}{4\pi\epsilon_o |r|^3} = \frac{Q'}{4\pi\epsilon_o r^2} a_r \quad (\text{V/m})$$

Note that the electric field points radially outward given a positive point charge at the origin and radially inward given a negative point charge at the origin. In either case, the electric field of the a point charge at the origin is spherically symmetric and easily defined using spherical coordinates. The magnitude of the point charge electric field varies as r^{-2}.

Postive charge ($Q>0$)   Negative charge ($Q<0$)
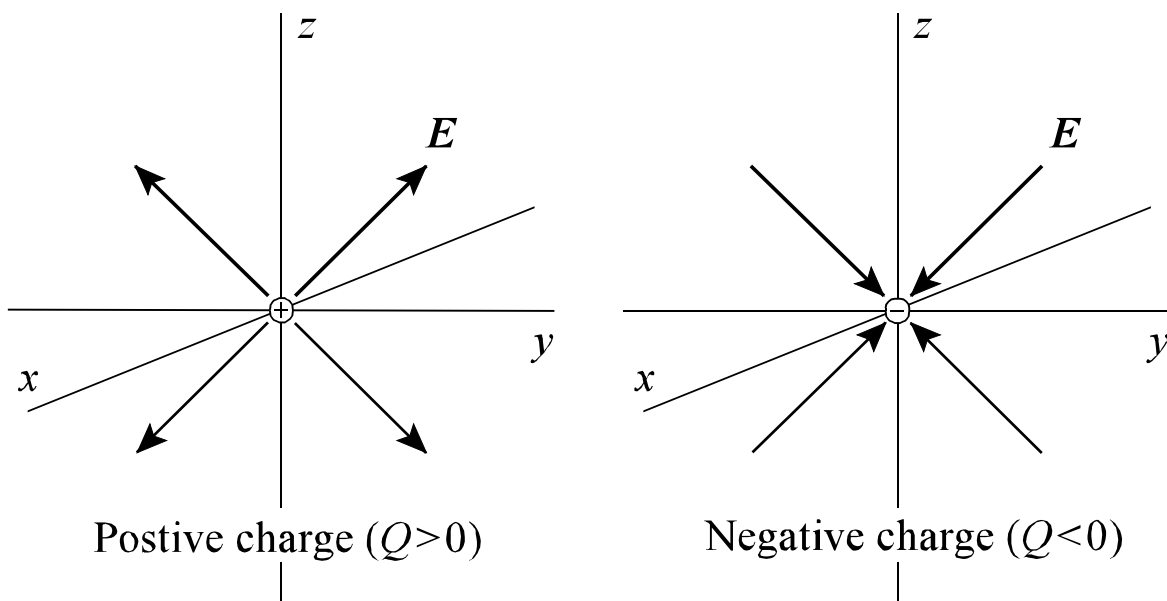
The vector force on a test charge Q at $\boldsymbol{r}$ due to a system of point charges $(Q_1{'}, Q_2{'},..., Q_N{'})$ at $(\boldsymbol{r}_1{'}, \boldsymbol{r}_2{'},..., \boldsymbol{r}_N{'})$ is, by superposition,

$$F(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_o} \sum_{k=1}^{N} Q_k{'} \frac{(\boldsymbol{r} - \boldsymbol{r}_k{'})}{|\boldsymbol{r} - \boldsymbol{r}_k{'}|^3}$$

The resulting electric field is

$$E(\boldsymbol{r}) \equiv \frac{F(\boldsymbol{r})}{Q} = \frac{1}{4\pi\epsilon_o} \sum_{k=1}^{N} Q_k{'} \frac{(\boldsymbol{r} - \boldsymbol{r}_k{'})}{|\boldsymbol{r} - \boldsymbol{r}_k{'}|^3}$$

<u>Example</u> (Electric field due to point charges)

Determine the vector electric field at $(1,-3,7)$ m due to point charges $Q_1{'} = 5$ nC at $(2,0,4)$ m and $Q_2{'} = -2$ nC at $(-3,0,5)$ m.

$$\boldsymbol{r} = \boldsymbol{a}_x - 3\boldsymbol{a}_y + 7\boldsymbol{a}_z$$

$$\boldsymbol{r}_1{'} = 2\boldsymbol{a}_x + 4\boldsymbol{a}_z$$
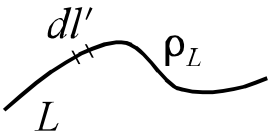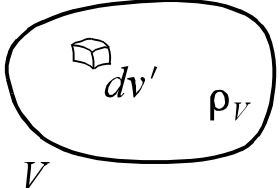
$$\boldsymbol{r}_2{'} = -3\boldsymbol{a}_x + 5\boldsymbol{a}_z$$

$$E(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_o} \left[Q_1{'} \frac{(\boldsymbol{r} - \boldsymbol{r}_1{'})}{|\boldsymbol{r} - \boldsymbol{r}_1{'}|^3} + Q_2{'} \frac{(\boldsymbol{r} - \boldsymbol{r}_2{'})}{|\boldsymbol{r} - \boldsymbol{r}_2{'}|^3} \right]$$

$$= \frac{1}{4\pi\epsilon_o} \left[(5\times10^{-9}) \frac{(-\boldsymbol{a}_x - 3\boldsymbol{a}_y + 3\boldsymbol{a}_z)}{[1^2 + 3^2 + 3^2]^{3/2}} \right.$$

$$\left. + (-2\times10^{-9}) \frac{(4\boldsymbol{a}_x - 3\boldsymbol{a}_y + 2\boldsymbol{a}_z)}{[4^2 + 3^2 + 2^2]^{3/2}} \right]$$

$$= 8.988 \left[\frac{5}{(19)^{3/2}} (-\boldsymbol{a}_x - 3\boldsymbol{a}_y + 3\boldsymbol{a}_z) - \frac{2}{(29)^{3/2}} (4\boldsymbol{a}_x - 3\boldsymbol{a}_y + 2\boldsymbol{a}_z) \right]$$

$$\boxed{E(\boldsymbol{r}) = (-1.004\,\boldsymbol{a}_x - 1.284\,\boldsymbol{a}_y + 1.399\,\boldsymbol{a}_z) \text{ (V/m)}}$$

Charge Distributions

Charges encountered in many electromagnetic applications (e.g., charged plates, wires, spheres, etc.) can be modeled as line, surface or volume charges. The electric field equation for a point charge can be extended to these charge distributions by viewing these distributions as simply a grouping of point charges.

Charge Distribution	Charge Density	Units	Total Charge	
point charge	Q	C	$Q_{total} = Q$	$\bullet \, Q$
line charge	ρ_L	C/m	$Q_{total} = \int_L \rho_L \, dl'$	
surface charge	ρ_S	C/m^2	$Q_{total} = \iint_S \rho_S \, ds'$	
volume charge	ρ_V	C/m^3	$Q_{total} = \iiint_V \rho_V \, dv'$	

In general, the various charge densities vary with position over the line, surface or volume and require an integration to determine the total charge associated with the charge distribution. *Uniform charge densities* do not vary with position and the total charge is easily determined as the product of the charge density and the total length, area or volume.

Uniform Charge Distributions

Uniform line charge (ρ_L = constant)

$$Q_{total} = \int_L \rho_L \, dl' = \rho_L \int_L dl' = \rho_L L_o \qquad \Rightarrow \qquad \rho_L = \frac{Q_{total}}{L_o}$$

(L_o = total line length)

Uniform surface charge (ρ_S = constant)

$$Q_{total} = \iint_S \rho_S \, ds' = \rho_S \iint_S ds' = \rho_S A_o \qquad \Rightarrow \qquad \rho_S = \frac{Q_{total}}{A_o}$$

(A_o = total surface area)

Uniform volume charge (ρ_V = constant)

$$Q_{total} = \iiint_V \rho_V \, dv' = \rho_V \iiint_V dv' = \rho_V V_o \qquad \Rightarrow \qquad \rho_V = \frac{Q_{total}}{V_o}$$

(V_o = total volume)

Electric Fields Due to Charge Distributions

Each differential element of charge on a line charge (dl'), a surface charge (ds') or a volume charge (dv') can be viewed as a point charge. By superposition, the total electric field produced by the overall charge distribution is the vector summation (integration) of the individual contributions due to each differential element. Using the equation for the electric field of a point charge, we can formulate an expression for dE (the incremental vector electric field produced by the given differential element of charge). We then integrate dE over the appropriate line, surface or volume over which the charge is distributed to determine the total electric field E at the field point P.

Point Charge

$$E(r) = \frac{Q}{4\pi\epsilon_o R^2} a_R$$

$$R = |r - r'| \qquad a_R = \frac{r - r}{|r - r'|}$$
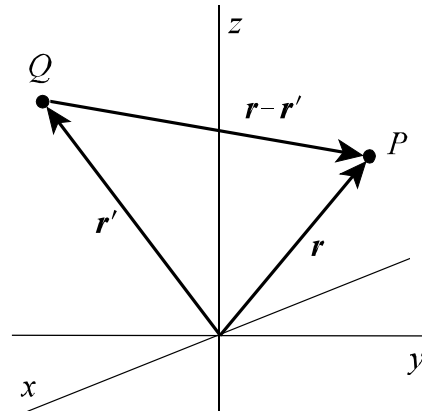
Line Charge $(\rho_L \, dl' \leftrightarrow Q)$

$$dE(r) = \frac{\rho_L \, dl'}{4\pi\epsilon_o R^2} a_R$$
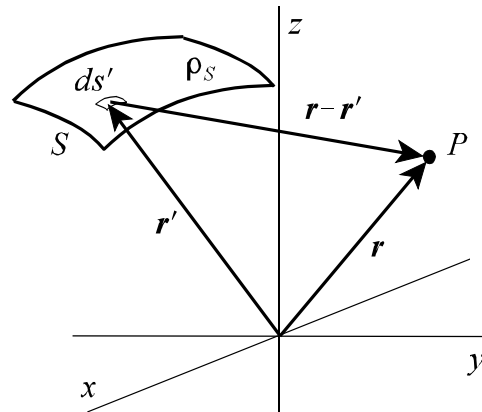
$$E(r) = \frac{1}{4\pi\epsilon_o} \int_L \frac{\rho_L}{R^2} a_R \, dl'$$
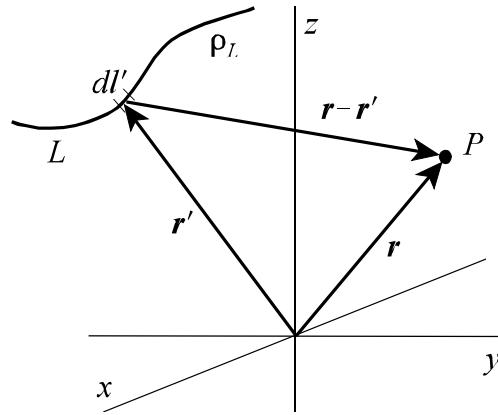
Surface Charge $(\rho_S \, ds' \leftrightarrow Q)$
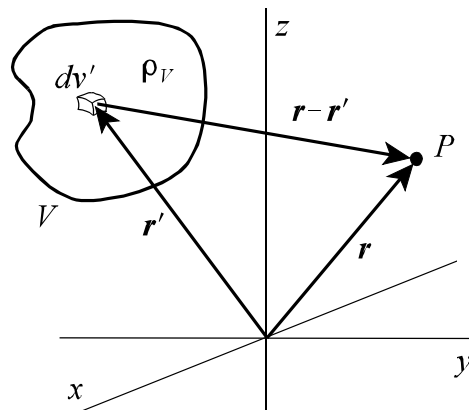
$$dE(r) = \frac{\rho_S \, ds'}{4\pi\epsilon_o R^2} a_R$$

$$E(r) = \frac{1}{4\pi\epsilon_o} \iint_S \frac{\rho_S}{R^2} a_R \, ds'$$

Volume Charge $(\rho_V \, dv' \leftrightarrow Q)$

$$dE(r) = \frac{\rho_V \, dv'}{4\pi\epsilon_o R^2} a_R$$

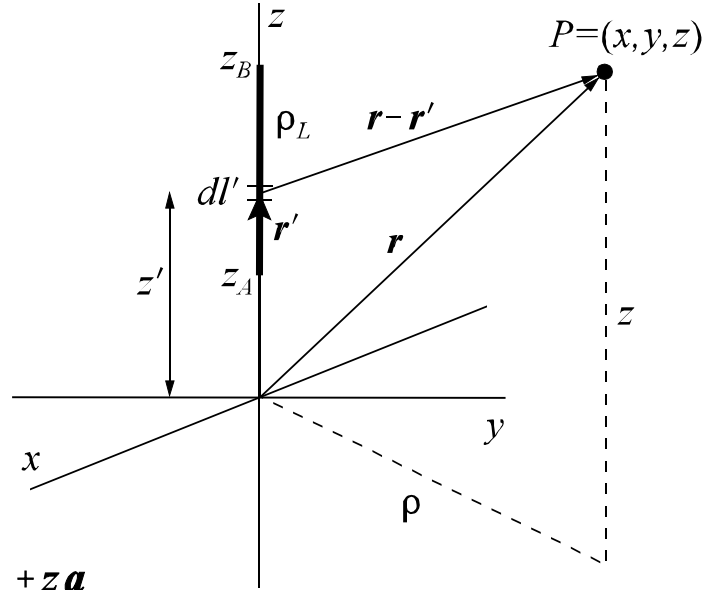$$E(r) = \frac{1}{4\pi\epsilon_o} \iiint_V \frac{\rho_V}{R^2} a_R \, dv'$$

<u>Example</u> (E due to a line charge)

Evaluate E at $P = (x, y, z)$ due to a uniform line charge lying along the z-axis between $(0, 0, z_A)$ and $(0, 0, z_B)$ with $z_B > z_A$.

$$E(r) = \frac{1}{4 \pi \epsilon_o} \int_L \frac{\rho_L}{R^2} \, a_R \, dl'$$

$$\rho_L = \frac{Q_{total}}{L} = \frac{Q_{total}}{z_B - z_A}$$

$$dl' = dz'$$

$$r = x a_x + y a_y + z a_z = \rho a_\rho + z a_z$$

$$r' = z' a_z$$

$$R = r - r' = \rho a_\rho + (z - z') a_z$$

$$R = |R| = \sqrt{\rho^2 + (z - z')^2}$$

$$a_R = \frac{R}{|R|} = \frac{\rho a_\rho + (z - z') a_z}{\sqrt{\rho^2 + (z - z')^2}}$$

$$E(r) = \frac{\rho_L}{4 \pi \epsilon_o} \int_{z_A}^{z_B} \frac{\rho a_\rho + (z - z') a_z}{[\rho^2 + (z - z')^2]^{3/2}} \, dz'$$

$$= \frac{\rho_L}{4 \pi \epsilon_o} \left[\rho a_\rho \int_{z_A}^{z_B} \frac{dz'}{[\rho^2 + (z - z')^2]^{3/2}} + a_z \int_{z_A}^{z_B} \frac{(z - z')}{[\rho^2 + (z - z')^2]^{3/2}} \, dz' \right]$$

The integrals in the electric field expression may be evaluated analytically using the following variable transformation:

$$\text{Let } \alpha = z - z' \qquad d\alpha = -dz'$$
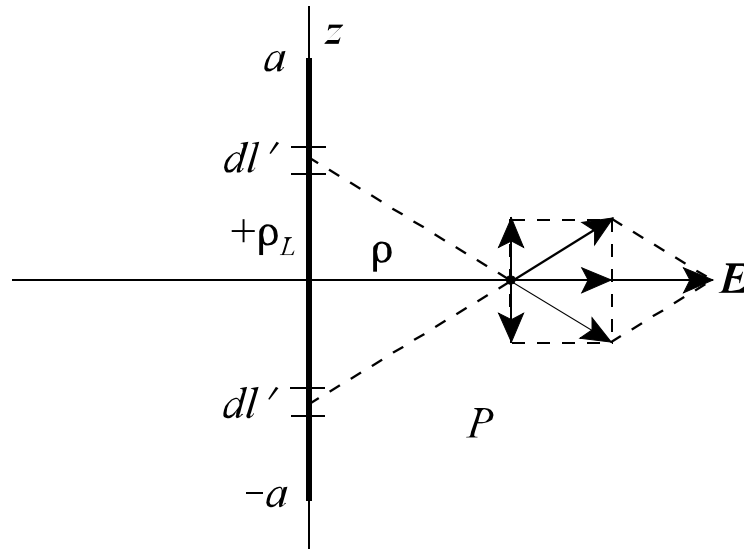
$$z' = z_A \qquad \rightarrow \qquad \alpha = z - z_A$$

$$z' = z_B \qquad \rightarrow \qquad \alpha = z - z_B$$

$$E(r) = \frac{\rho_L}{4\pi\epsilon_o}\left[-\rho\,a_\rho \int_{z-z_A}^{z-z_B} \frac{d\alpha}{(\rho^2 + \alpha^2)^{3/2}} - a_z \int_{z-z_A}^{z-z_B} \frac{\alpha\,d\alpha}{(\rho^2 + \alpha^2)^{3/2}} \right]$$

$$\int \frac{dx}{(\rho^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2 + a^2}} \qquad \int \frac{x\,dx}{(\rho^2 + x^2)^{3/2}} = -\frac{1}{\sqrt{x^2 + a^2}}$$

$$E(r) = \frac{\rho_L}{4\pi\epsilon_o}\left\{ -\rho\,a_\rho \left[\frac{\alpha}{\rho^2\sqrt{\alpha^2 + \rho^2}} \right]_{z-z_A}^{z-z_B} + a_z \left[\frac{1}{\sqrt{\alpha^2 + \rho^2}} \right]_{z-z_A}^{z-z_B} \right\}$$

$$= \frac{\rho_L}{4\pi\epsilon_o\rho}\left\{ \left[\frac{z-z_A}{\sqrt{\rho^2 + (z-z_A)^2}} - \frac{z-z_B}{\sqrt{\rho^2 + (z-z_B)^2}} \right]a_\rho \right.$$

$$\left. + \left[\frac{\rho}{\sqrt{\rho^2 + (z-z_B)^2}} - \frac{\rho}{\sqrt{\rho^2 + (z-z_A)^2}} \right]a_z \right\}$$

For the special case of a line charge centered at the coordinate origin ($z_A = -a$, $z_B = a$) with the field point P lying in the x-y plane [$P = (x,y,0)$], the electric field expression reduces to

$$E(r) = \frac{\rho_L}{4\pi\epsilon_o\rho}\left\{\left[\frac{a}{\sqrt{\rho^2+a^2}} - \frac{-a}{\sqrt{\rho^2+a^2}}\right]a_\rho \right.$$

$$\left. + \left[\frac{\rho}{\sqrt{\rho^2+a^2}} - \frac{\rho}{\sqrt{\rho^2+a^2}}\right]a_z\right\}$$

$$E(r) = \frac{\rho_L a}{2\pi\epsilon_o\rho\sqrt{\rho^2+a^2}}\,a_\rho$$

(E-field in the *x-y* plane due to a uniform line charge of length 2a centered at the origin)

To determine the electric field of an infinite length line charge, we take the limit of the previous result as *a* approaches ∞.

$$E(r) = \frac{\rho_L}{2\pi\epsilon_o\rho}\,a_\rho \lim_{a\to\infty}\left[\frac{a}{\sqrt{\rho^2+a^2}}\right]$$

$$= \frac{\rho_L}{2\pi\epsilon_o\rho}\,a_\rho \lim_{a\to\infty}\left[\frac{1}{\sqrt{(\rho/a)^2+1}}\right]$$

$$E(r) = \frac{\rho_L}{2\pi\epsilon_o\rho}\,a_\rho$$

(E-field due to a uniform line charge of infinite length lying along the *z*-axis.)

Note that the electric field of the infinite-length uniform line charge is cylindrically symmetric (*line source*). That is, the electric field is independent of ϕ due to the symmetry of the source. The electric field of the infinite-length uniform line charge is also independent of z due to the infinite length of the uniform source. In comparison to the electric field of a point charge (which varies as r^{-2}), the electric field of the infinite-length uniform line charge varies as ρ^{-1}. If ρ_L is positive, the electric field points outward radially while a negative ρ_L produces an electric field which points inward radially.
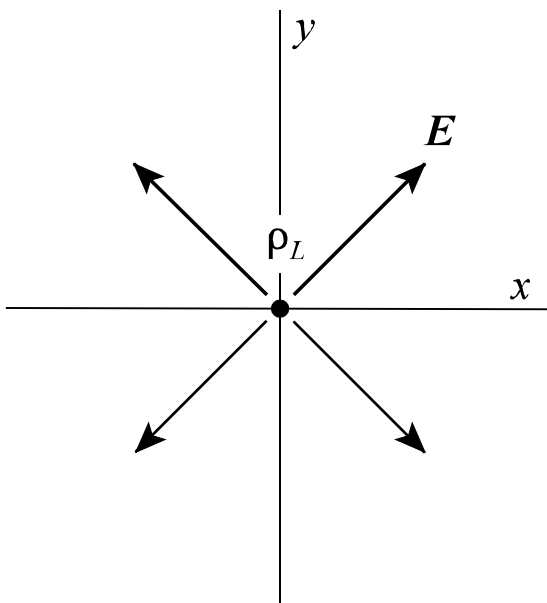
Infinite-length uniform
line charge ($\rho_L > 0$)
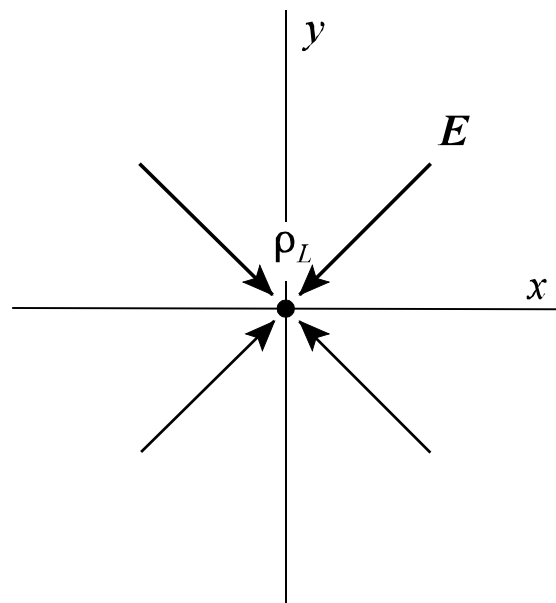
Infinite-length uniform
line charge ($\rho_L < 0$)

<u>Example</u> (E due to a surface charge)

Evaluate E at a point on the z-axis $P = (0,0,h)$ due to a uniformly charged disk of radius a lying in the x-y plane and centered at the coordinate origin.

$$E(r) = \frac{1}{4\pi\epsilon_o} \iint_S \frac{\rho_S}{R^2} a_R \, ds'$$

$$\rho_S = \frac{Q_{total}}{A} = \frac{Q_{total}}{\pi a^2}$$

$$ds' = \rho' d\rho' d\phi'$$

$$r = h a_z$$

$$r' = \rho' a_\rho$$

$$R = r - r' = h a_z - \rho' a_\rho$$

$$R = |R| = \sqrt{\rho'^2 + h^2}$$

$$a_R = \frac{R}{|R|} = \frac{-\rho' a_\rho + h a_z}{\sqrt{\rho'^2 + h^2}}$$

$$E(r) = \frac{\rho_S}{4\pi\epsilon_o} \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^{a} \frac{-\rho' a_\rho + h a_z}{(\rho'^2 + h^2)^{3/2}} \rho' d\rho' d\phi'$$
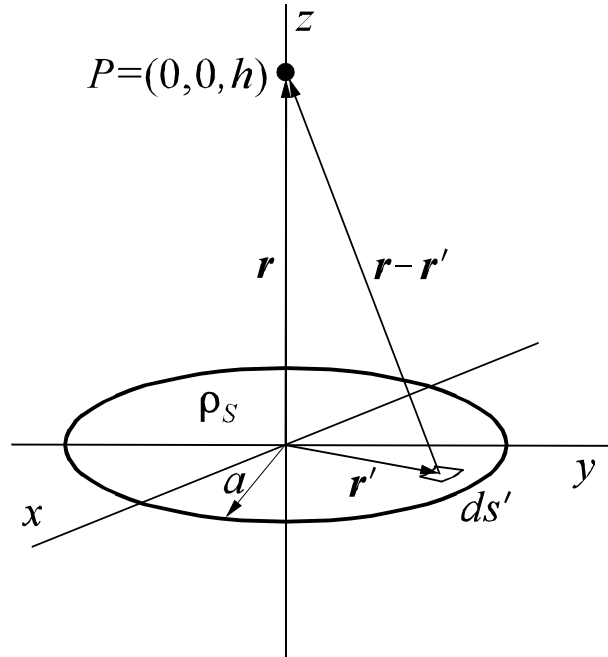
The unit vector a_ρ, which is a function of the integration variable ϕ', can be transformed into rectangular coordinate unit vectors to simplify the integration.

$$a_\rho = \cos\phi' a_x + \sin\phi' a_y$$

$$E(r) = \frac{\rho_S}{4\pi\epsilon_o} \left\{ a_x \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^{a} \frac{-\rho'^2 \cos\phi'}{(\rho'^2+h^2)^{3/2}} d\rho' d\phi' \right.$$

$$+ a_y \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^{a} \frac{-\rho'^2 \sin\phi'}{(\rho'^2+h^2)^{3/2}} d\rho' d\phi'$$

$$\left. + a_z h \int_{\phi'=0}^{2\pi} \int_{\rho'=0}^{a} \frac{\rho'}{(\rho'^2+h^2)^{3/2}} d\rho' d\phi' \right\}$$

The first two integrals in the electric field expression are zero given the sine and cosine integrals with respect to ϕ' over one period.

$$\int_0^{2\pi} \sin\phi' d\phi' = \int_0^{2\pi} \cos\phi' d\phi' = 0$$

The electric field expression reduces to

$$E(r) = \frac{\rho_S h}{4\pi\epsilon_o} a_z \int_0^{2\pi} d\phi' \int_0^a \frac{\rho' d\rho'}{(\rho'^2+h^2)^{3/2}} = \frac{\rho_S h}{4\pi\epsilon_o} a_z [\phi']_0^{2\pi} \left[\frac{-1}{\sqrt{\rho'^2+h^2}} \right]_0^a$$

$$= \frac{\rho_S h}{2\epsilon_o} \left[\frac{1}{h} - \frac{1}{\sqrt{a^2+h^2}} \right] a_z$$

(E-field on the z-axis due to a uniformly charged disk of radius a in the x-y plane centered at the origin, h = height above disk)

The electric field produced by an infinite charged sheet can be determined by taking the limit of the charged disk E as the disk radius approaches ∞.
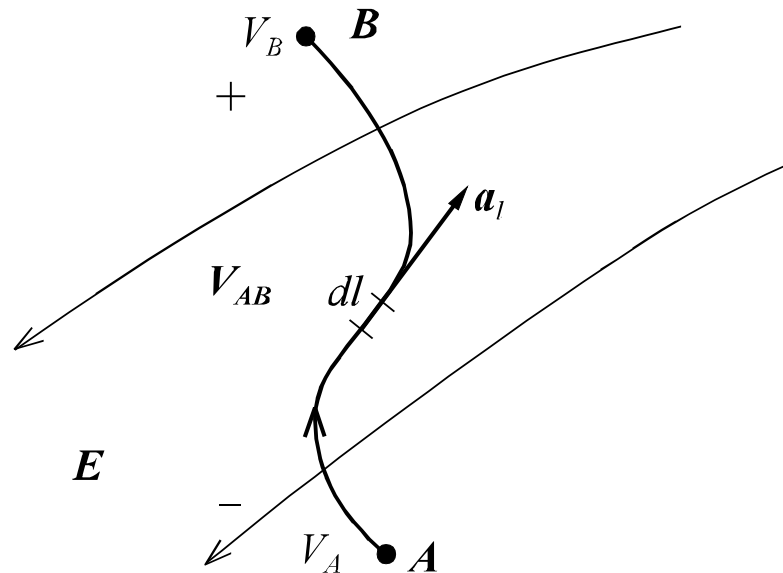
$$E(\text{infinite sheet}) = \lim_{a\to\infty} \left[E(\text{disk, radius}=a) \right]$$

$$E(r) = \frac{\rho_S h}{2\epsilon_o} a_z \lim_{a\to\infty} \left[\frac{1}{h} - \frac{1}{\sqrt{a^2+h^2}} \right] = \frac{\rho_S}{2\epsilon_o} a_z$$

(E-field due to a uniformly charged infinite sheet)

Note that the electric field of the uniformly charged infinite sheet is uniform (independent of the height h of the field point above the sheet).

Electric Scalar Potential

Given that the electric field defines the force per unit charge acting on a positive test charge, any attempt to move the test charge against the electric field requires that work be performed. The *potential difference* between two points in an electric field is defined as the work per unit charge performed when moving a positive test charge from one point to the other.

From Coulomb's law, the vector force on a positive point charge in an electric field is given by

$$\boldsymbol{F} = Q\boldsymbol{E}$$

The amount of work performed in moving this point charge in the electric field is product of the force and the distance moved. When the positive point charge is moved against the force (against the electric field), the work done is positive. When the point charge is moved in the direction of the force, the work done is negative. If the point charge is moved in a direction perpendicular to the force, the amount of work done is zero. For a differential element of length (*dl*), the small amount of work done (*dW*) is defined as

$$dW = -\boldsymbol{F} \cdot d\boldsymbol{l} = -Q\boldsymbol{E} \cdot d\boldsymbol{l}$$

The minus sign in the previous equation is necessary to obtain the proper sign on the work done (positive when moving the test charge against the electric field). When the point charge is moved along a path from point A to B, the total amount of work performed (W) is found by integrating dW along the path.

$$W = \int_A^B dW = -\int_A^B \boldsymbol{F} \cdot d\boldsymbol{l} = -Q \int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} \qquad \text{(J)}$$

The potential difference between A and B is then

$$V_{AB} = \frac{W}{Q} = -\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} \qquad \left(\frac{\text{J}}{\text{C}} = \text{V} \right)$$

The potential difference equation may be written as

$$V_{AB} = -\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} = \int_A^B dV = V_B - V_A$$

where V_A and V_B are the *absolute potentials* at points A and B, respectively. The absolute potential at a point is defined as the potential difference between the point and a reference point an infinite distance away. The definition of the potential difference in terms of the absolute potentials at the starting and ending points of the path shows that the potential difference between any two points is independent of the path taken between the points.

For a closed path (point A = point B), the line integral of the electric field yields the potential difference between a point and itself yielding a value of zero.

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

Vector fields which have zero-valued closed path line integrals are designated as *conservative* fields. All electrostatic fields are conservative fields.

Example (Potential difference)

Determine the absolute potential in the electric field of a point charge Q located at the coordinate origin.

The electric field of a point charge at the origin is

$$E = \frac{Q}{4\pi\epsilon_o r^2}\, a_r$$

The potential difference between two points A and B in the electric field of the point charge is

$$V_{AB} = -\int_A^B E\cdot dl = \int_A^B dV = V_B - V_A$$

If we choose an inward radial path from $r = r_A$ to $r = r_B$, the vector differential length is

$$dl = dl\,a_l = (-dr)(-a_r) = dr\,a_r$$

which yields

$$V_{AB} = -\int_{r_A}^{r_B}\left(\frac{Q}{4\pi\epsilon_o r^2}\,a_r\right)\cdot(dr\,a_r) = -\frac{Q}{4\pi\epsilon_o}\int_{r_A}^{r_B}\frac{dr}{r^2}$$

$$= -\frac{Q}{4\pi\epsilon_o}\left[-\frac{1}{r}\right]_{r_A}^{r_B} = \frac{Q}{4\pi\epsilon_o}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

The absolute potential at point B is found by taking the limit as r_A approaches infinity.

$$V_B = \lim_{r_A\to\infty} V_{AB} = \frac{Q}{4\pi\epsilon_o}\lim_{r_A\to\infty}\left[\frac{1}{r_B} - \frac{1}{r_A}\right] = \frac{Q}{4\pi\epsilon_o r_B}$$

Potentials of Charge Distributions

The previous formula can be generalized as the absolute potential of a point charge at the origin (let $r_B = r$).

$$V = \frac{Q}{4\pi\epsilon_o r}$$

(Absolute potential for a point charge at the origin)

Note that the potential distribution of the point charge exhibits spherical symmetry just like the electric field. The potential of the point charge varies as r^{-1} in comparison to the electric field of a point charge which varies as r^{-2}. Surfaces on which the potential is constant are designated as *equipotential surfaces*. Equipotential surfaces are always perpendicular to the electric field (since no work is performed to move a charge perpendicular to the electric field). For the point charge, the equipotential surfaces are concentric spherical surfaces about the point charge.

The absolute potential of a point charge at an arbitrary location is

$$V = \frac{Q}{4\pi\epsilon_o |r - r'|}$$

(Absolute potential for a point charge at an arbitrary location)

The principle of superposition can be applied to the determine the potential due to a set of point charges which yields

$$V = \frac{1}{4\pi\epsilon_o} \sum_{k=1}^{N} \frac{Q_k}{|r - r'_k|}$$

(Absolute potential of a set of point charges)

The potentials due to line, surface and volume distributions of charge are found by integrating the incremental potential contribution due to each differential element of charge in the distribution.

Point Charge

$$V(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_o R}$$

$$R = \left| \boldsymbol{r} - \boldsymbol{r}' \right|$$

Line Charge $(\rho_L \, dl' \leftrightarrow Q)$

$$dV(\boldsymbol{r}) = \frac{\rho_L \, dl'}{4\pi\epsilon_o R}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_o} \int_L \frac{\rho_L}{R} \, dl'$$

Surface Charge $(\rho_S \, ds' \leftrightarrow Q)$

$$dV(\boldsymbol{r}) = \frac{\rho_S \, ds'}{4\pi\epsilon_o R}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_o} \iint_S \frac{\rho_S}{R} \, ds'$$

Volume Charge $(\rho_V \, dv' \leftrightarrow Q)$

$$dV(\boldsymbol{r}) = \frac{\rho_V \, dv'}{4\pi\epsilon_o R}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_o} \iiint_V \frac{\rho_V}{R} \, dv'$$

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Example (Potential due to a line charge)

Determine the potential in the *x-y* plane due to a uniform line charge of length $2a$ lying along the *z*-axis and centered at the coordinate origin

$$V = \frac{1}{4\pi\epsilon_o}\int_L \frac{\rho_L}{R}\,dl'$$

$$dl' = dz'$$

$$r = \rho\,a_\rho$$

$$r' = z'\,a_z$$

$$R = r - r' = \rho\,a_\rho - z'\,a_z$$

$$R = |R| = \sqrt{z'^2 + \rho^2}$$

$$V = \frac{\rho_L}{4\pi\epsilon_o}\int_{-a}^{a}\frac{dz'}{\sqrt{z'^2+\rho^2}} = (2)\frac{\rho_L}{4\pi\epsilon_o}\int_{0}^{a}\frac{dz'}{\sqrt{z'^2+\rho^2}}$$

Even integrand
Symmetric limits

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \ln[x+\sqrt{x^2+a^2}]$$

$$V = \frac{\rho_L}{2\pi\epsilon_o}\ln[z'+\sqrt{z'^2+\rho^2}]\Big|_0^a = \frac{\rho_L}{2\pi\epsilon_o}\left\{\ln[a+\sqrt{a^2+\rho^2}] - \ln\rho\right\}$$

$$= \frac{\rho_L}{2\pi\epsilon_o}\ln\left[\frac{a+\sqrt{\rho^2+a^2}}{\rho}\right]$$

(Absolute potential in the *x-y* plane due to a uniform line charge of length $2a$ lying along

the z-axis centered at the coordinate origin)

<u>Example</u> (Potential due to a square loop)

Determine the potential at the center of a square loop of side length l which is uniformly charged.

The uniformly charged square loop can be viewed as four line charges. The total potential at the center of the loop is the scalar sum of the contributions from the four sides (identical scalar contributions). Thus, the potential at P due to one side of the loop is
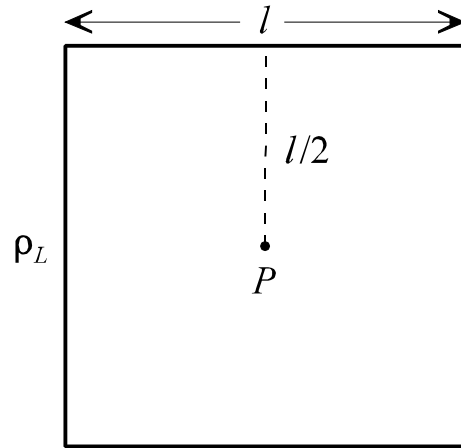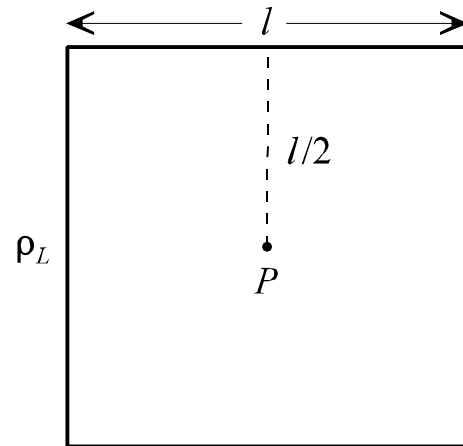
$$V_{side} = \frac{\rho_L}{2\pi\epsilon_o} \ln\left[\frac{a + \sqrt{\rho^2 + a^2}}{\rho}\right] \qquad a = \frac{l}{2} \qquad \rho = \frac{l}{2}$$

$$V_{total} = 4\,V_{side} = (4)\,\frac{\rho_L}{2\pi\epsilon_o} \ln\left[\frac{l/2 + \sqrt{(l/2)^2 + (l/2)^2}}{l/2}\right]$$

$$= \frac{2\rho_L}{\pi\epsilon_o} \ln\left[1 + \sqrt{2}\,\right]$$

Electric Field as the Gradient of the Potential

The potential difference between two points in an electric field can be written as the line integral of the electric field such that

$$V_{AB} = -\int_A^B \boldsymbol{E} \cdot \boldsymbol{dl} = \int_A^B dV = V_B - V_A$$

From the equation above, the incremental change in potential along the integral path is

$$dV = -\boldsymbol{E} \cdot \boldsymbol{dl} = -\boldsymbol{E} \cdot \boldsymbol{a}_l\, dl = -E\cos\theta\, dl$$

where θ is the angle between the direction of the integral path and the electric field. The derivative of the potential with respect to position along the path may be written as

$$\frac{dV}{dl} = -E\cos\theta$$

Note that the potential derivative is a maximum when $\theta = \pi$ (when the direction of the electric field is opposite to the direction of the path). Thus,

$$\left[\frac{dV}{dl}\right]_{\mathbf{max}} = E \qquad \text{when } \theta = \pi \quad (\cos\theta = -1)$$

This equation shows that the magnitude of the electric field is equal to the maximum space rate of change in the potential. The direction of the electric field is the direction of the maximum decrease in the potential (the electric field always points from a region of higher potential to a region of lower potential).

The electric field can be written in terms of the potential as

$$E = \frac{dV}{dl}(-a_l) = -\frac{dV}{dl}a_l = -\nabla V$$

where the operator "∇" (del) is the *gradient operator.* The gradient operator is a differential operator which operates on a scalar function to yield (1) the maximum increase per unit distance and (2) the direction of the maximum increase. Since the electric field always points in the direction of decreasing potential, the electric field is the negative of the gradient of *V*.

The derivative with respect to *l* in the gradient operator above can be generalized to a particular coordinate system by including the variation in the potential with respect to the three coordinate variables. In rectangular coordinates,

$$\nabla V = \frac{dV}{dl}a_l = \frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z$$

$$\nabla = \frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z \qquad \text{(Gradient operator in rectangular coordinates)}$$

The gradient operator is defined differently in rectangular, cylindrical and spherical coordinates. The electric field expression as the gradient of the potential in these coordinate systems are

$$E = -\nabla V = -\left[\frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z\right] \qquad (\text{rectangular})$$

$$= -\left[\frac{\partial V}{\partial \rho}a_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}a_\phi + \frac{\partial V}{\partial z}a_z\right] \qquad (\text{cylindrical})$$

$$= -\left[\frac{\partial V}{\partial r}a_r + \frac{1}{r}\frac{\partial V}{\partial \theta}a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}a_\phi\right] \qquad (\text{spherical})$$

<u>Example</u> (E as the gradient of V)

Given $V(r,\theta,\phi) = \dfrac{10}{r^2}\sin\theta\cos\phi$ (a.) find $E(r,\theta,\phi)$ and (b.) E at $(2,\pi/2,0)$.

(a.)

$$E = -\nabla V = -\left[\frac{\partial V}{\partial r}\,a_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\,a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\,a_\phi\right]$$

$$= -\left[10\sin\theta\cos\phi\,\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)a_r + \frac{10}{r^3}\cos\phi\,\frac{\partial}{\partial\theta}(\sin\theta)\,a_\theta\right.$$

$$\left.+ \frac{10}{r^3}\frac{\partial}{\partial\phi}(\cos\phi)\,a_\phi\right]$$

$$= \frac{20}{r^3}\sin\theta\cos\phi\,a_r - \frac{10}{r^3}\cos\theta\cos\phi\,a_\theta + \frac{10}{r^3}\sin\phi\,a_\phi$$

$$= E_r a_r + E_\theta a_\theta + E_\phi a_\phi$$

(b.)

$$E(2,\pi/2,0) = \frac{20}{8}\,a_r + 0\,a_\theta + 0\,a_\phi = \frac{20}{8}\,a_r\ (V/m)$$

Summary of Electric Field / Potential Relationships

$$V = -\int E\cdot dl \qquad\qquad \text{Integrate } E \text{ to find } V$$

$$E = -\nabla V \qquad\qquad \text{Differentiate } V \text{ to find } E$$

Electric Flux Density

The *electric flux density* D in free space is defined as the product of the free space permittivity (ϵ_o) and the electric field (E):

$$D = \epsilon_o E$$

Given that the electric field is inversely proportional to the permittivity of the medium, the electric flux density is independent of the medium properties.

point charge	$E = \dfrac{Q}{4\pi\epsilon_o r^2} \, a_r$	$D = \dfrac{Q}{4\pi r^2} \, a_r$
line charge	$E = \dfrac{\rho_L}{2\pi\epsilon_o \rho} \, a_\rho$	$D = \dfrac{\rho_L}{2\pi \rho} \, a_\rho$

The units on electric flux density are

$$\frac{F}{m} \times \frac{V}{m} = \frac{C}{m^2}$$

so that the units on electric flux density are equivalent to surface charge density.

The *total electric flux* (ψ) passing through a surface S is defined as the integral of the normal component of D through the surface.

$$\psi = \iint_S D \cdot ds = \iint_S D \cdot a_n \, ds = \iint_S D_n \, ds$$

where a_n is the unit normal to the surface S and D_n is the component of D normal to S. The direction chosen for the unit normal (one of two possible) defines the direction of the total flux.

For a closed surface, the total electric flux is

$$\psi = \oint_S D \cdot ds$$

a_n – outward (total outward flux)

a_n – inward (total inward flux)

Gauss's Law

Gauss's law is one of the set of four *Maxwell's equations* that govern the behavior of electromagnetic fields.

Gauss's Law - *The total outward electric flux ψ through any closed surface is equal to the total charge enclosed by the surface.*

Gauss's law is written in equation form as

$$\psi = \oint_S \mathbf{D} \cdot \mathbf{ds} = Q_{enclosed} \qquad (\text{Gauss's law})$$

where $\mathbf{ds} = \mathbf{a}_n \, ds$ and $\mathbf{a}_n$ is the *outward* pointing unit normal to *S*.

<u>Example</u> (Gauss's law, point charge at origin)

Given a point charge at the origin, show that Gauss's law is valid on a spherical surface (*S*) of radius r_o.

Gauss's law applied to the spherical surface *S* surrounding the point charge *Q* at the origin should yield

$$\psi = \oint_S \mathbf{D} \cdot \mathbf{ds} = Q_{enclosed} = Q$$

The electric flux produced by *Q* is

$$\mathbf{D} = \frac{Q}{4\pi r^2} \mathbf{a}_r$$

On the spherical surface S of radius r_o, we have

$$D(r=r_o) = \frac{Q}{4\pi r_o^2}\, a_r \qquad\qquad ds = r_o^2 \sin\theta\, d\theta\, d\phi\, a_r$$

$$D \cdot ds = \left(\frac{Q}{4\pi r_o^2}\, a_r\right) \cdot \left(r_o^2 \sin\theta\, d\theta\, d\phi\, a_r\right) = \frac{Q}{4\pi}\sin\theta\, d\theta\, d\phi$$

$$\psi = \oint_S D \cdot ds = \frac{Q}{4\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin\theta\, d\theta\, d\phi = \frac{Q}{4\pi}\left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= \frac{Q}{4\pi}(2)(2\pi) = Q \qquad (\text{charge enclosed})$$

Note the **outward** pointing normal requirement in Gauss's law is a direct result of our electric field (flux) convention.

> interior positive charges → outward electric flux
>
> interior negative charges → inward electric flux

By using an outward pointing normal, we obtain the correct sign on the enclosed charge.

Gauss's law can also be used to determine the electric fields produced by simple charge distributions that exhibit special symmetry. Examples of such charge distributions include uniformly charged spherical surfaces and volumes.

Example (Using Gauss's law to determine E)

Use Gauss's law to determine the vector electric field inside and outside a uniformly charged spherical volume of radius a.

$$\rho_V = \begin{cases} k & r < a \\ 0 & r > a \end{cases}$$

$$k = \text{constant}$$

S – spherical surface of radius $r = a$

S_+ – spherical surface of radius $r > a$

S_- – spherical surface of radius $r < a$

Gauss's law can be applied on S_- to determine the electric field inside the charged sphere [$E(r<a)$].

$$\oint_{S_-} \boldsymbol{D} \cdot \boldsymbol{ds} = Q_{enclosed} = \oint_{V_-} \rho_V \, dv$$

$$(V_- = \text{volume enclosed by } S_-)$$

$$\boldsymbol{ds} = ds \, \boldsymbol{a}_r$$

$$\oint_{S_-} \boldsymbol{D} \cdot \boldsymbol{ds} = \oint_{S_-} \boldsymbol{D} \cdot \boldsymbol{a}_r \, ds$$

$$= \oint_{S_-} D_r \, ds = Q_{enclosed}$$

By symmetry, on S_- (and S_+), D_r is uniform and has only an $\boldsymbol{a}_r$ component.

$$\oint_{S_-} D_r\, ds = D_r \oint_{S_-} ds = D_r (4\pi r^2) = Q_{enclosed}$$

$$= \oint_{V_-} \rho_V\, dv = \rho_V \oint_{V_-} dv = k\left(\frac{4}{3}\pi r^3\right)$$

$$D_r = \frac{k\left(\dfrac{4}{3}\pi r^3\right)}{4\pi r^2} = \frac{kr}{3} \qquad (r<a)$$

or

$$D = \frac{kr}{3}\, a_r \qquad E = \frac{D}{\epsilon_o} = \frac{kr}{3\epsilon_o}\, a_r \qquad (r<a)$$

Gauss's law can be applied on S_+ to determine the electric field outside the charged sphere [$E(r>a)$].

$$\oint_{S_+} D_r\, ds = D_r \oint_{S_+} ds = D_r (4\pi r^2) = Q_{enclosed}$$

$$= \oint_{V_+} \rho_V\, dv = \rho_V \oint_{V} dv = k\left(\frac{4}{3}\pi a^3\right)$$

$$(V = \text{volume enclosed by } S)$$

$$D_r = \frac{k\left(\dfrac{4}{3}\pi a^3\right)}{4\pi r^2} = \frac{ka^3}{3r^2} \qquad (r>a)$$

or

$$D = \frac{ka^3}{3r^2}\, a_r \qquad E = \frac{ka^3}{3\epsilon_o r^2}\, a_r \qquad (r>a)$$

$|E|$

$\dfrac{kr}{3\,\epsilon_o}$

$\dfrac{ka}{3\,\epsilon_o}$

Electric Field for the uniformly charged spherical volume of radius a

$\dfrac{ka^3}{3\,\epsilon_o r^2}$

a

r

Divergence Operator / Gauss's Law (Differential Form)

The differential form of Gauss's law is determined by applying the integral form of Gauss's law to a differential volume (Δv). The differential form of Gauss's law is defined in terms of the *divergence operator*. The divergence operator is obtained by taking the limit as Δv shrinks to zero (to the point P) of the flux out of Δv divided by Δv.

$\Delta v = \Delta x \Delta y \Delta z$

$P = (x_o, y_o, z_o)$

$$\oint_{\Delta v} \boldsymbol{D} \cdot \boldsymbol{ds} \qquad (\text{net flux out of } \Delta v)$$

$$\text{Div } \boldsymbol{D} = \nabla \cdot \boldsymbol{D} \equiv \lim_{\Delta v \to 0} \left[\frac{\oint_{\Delta v} \boldsymbol{D} \cdot \boldsymbol{ds}}{\Delta v} \right] = \lim_{\Delta v \to 0} \left[\frac{Q_{enclosed}}{\Delta v} \right] = \rho_V(P)$$

Gradient operator

$$\oint_S \boldsymbol{D} \cdot \boldsymbol{ds} = Q_{enclosed} \qquad \text{Gauss's law (integral form)}$$

$$\nabla \cdot \boldsymbol{D} = \rho_V \qquad \text{Gauss's law (differential form)}$$

The divergence operator in rectangular coordinates can be determined by performing the required integrations. The electric flux density within the differential volume is defined by

$$\mathbf{D} = D_x \mathbf{a}_x + D_y \mathbf{a}_y + D_z \mathbf{a}_z$$

while the electric flux density evaluated at the point P is defined as

$$\mathbf{D}(P) = D_{xo} \mathbf{a}_x + D_{yo} \mathbf{a}_y + D_{zo} \mathbf{a}_z$$

The total flux out of the differential volume Δv is

$$\underset{\Delta v}{\oint} \mathbf{D} \cdot d\mathbf{s} = \underset{\substack{front \\ face}}{\iint} \mathbf{D} \cdot d\mathbf{s} + \underset{\substack{back \\ face}}{\iint} \mathbf{D} \cdot d\mathbf{s} + \underset{\substack{left \\ face}}{\iint} \mathbf{D} \cdot d\mathbf{s} + \underset{\substack{right \\ face}}{\iint} \mathbf{D} \cdot d\mathbf{s} + \underset{\substack{top \\ face}}{\iint} \mathbf{D} \cdot d\mathbf{s} + \underset{\substack{bottom \\ face}}{\iint} \mathbf{D} \cdot d\mathbf{s}$$

The electric flux density components can be written in terms of a Taylor series about the point P.

$$D_x(x) = D_{xo} + \frac{\partial D_x}{\partial x}(x - x_o) + \frac{\partial^2 D_x}{\partial x^2} \frac{(x - x_o)^2}{2!} + \ldots$$

$$D_y(y) = D_{yo} + \frac{\partial D_y}{\partial y}(y - y_o) + \frac{\partial^2 D_y}{\partial y^2} \frac{(y - y_o)^2}{2!} + \ldots$$

$$D_z(z) = D_{zo} + \frac{\partial D_z}{\partial z}(z - z_o) + \frac{\partial^2 D_z}{\partial z^2} \frac{(z - z_o)^2}{2!} + \ldots$$

For points close to P (such as the faces on the differential volume), the higher order terms in the Taylor series expansions become negligible such that

$$D_x(x) \approx D_{xo} + \frac{\partial D_x}{\partial x}(x - x_o)$$

$$D_y(y) \approx D_{yo} + \frac{\partial D_y}{\partial y}(y - y_o)$$

$$D_z(z) \approx D_{zo} + \frac{\partial D_z}{\partial z}(z - z_o)$$

The flux densities on the six faces of the differential volume are

<u>front face</u>

$$D_x(x_o+\Delta x/2) \approx D_{xo} + \frac{\partial D_x}{\partial x}\frac{\Delta x}{2}$$

<u>back face</u>

$$D_x(x_o-\Delta x/2) \approx D_{xo} - \frac{\partial D_x}{\partial x}\frac{\Delta x}{2}$$

<u>right face</u>

$$D_y(y_o+\Delta y/2) \approx D_{yo} + \frac{\partial D_y}{\partial y}\frac{\Delta y}{2}$$

<u>left face</u>

$$D_y(y_o-\Delta y/2) \approx D_{yo} - \frac{\partial D_y}{\partial y}\frac{\Delta y}{2}$$

<u>top face</u>

$$D_z(z_o+\Delta z/2) \approx D_{zo} + \frac{\partial D_z}{\partial z}\frac{\Delta z}{2}$$

<u>bottom face</u>

$$D_z(z_o-\Delta z/2) \approx D_{zo} - \frac{\partial D_z}{\partial z}\frac{\Delta z}{2}$$

The integrations over the six sides of the differential volume yield

$$\iint_{\substack{front \\ face}} \boldsymbol{D}\cdot d\boldsymbol{s} + \iint_{\substack{back \\ face}} \boldsymbol{D}\cdot d\boldsymbol{s} = \int_0^{\Delta z} \int_0^{\Delta y} \left[D_{xo} + \frac{\partial D_x}{\partial x}\frac{\Delta x}{2} \right] \boldsymbol{a}_x\cdot dy\,dz\,\boldsymbol{a}_x$$

$$+ \int_0^{\Delta z} \int_0^{\Delta y} \left[D_{xo} - \frac{\partial D_x}{\partial x}\frac{\Delta x}{2} \right] \boldsymbol{a}_x\cdot dy\,dz\,(-\boldsymbol{a}_x)$$

$$\iint_{\substack{right \\ face}} \boldsymbol{D}\cdot d\boldsymbol{s} + \iint_{\substack{left \\ face}} \boldsymbol{D}\cdot d\boldsymbol{s} = \int_0^{\Delta z} \int_0^{\Delta x} \left[D_{yo} + \frac{\partial D_y}{\partial y}\frac{\Delta y}{2} \right] \boldsymbol{a}_y\cdot dx\,dz\,\boldsymbol{a}_y$$

$$+ \int_0^{\Delta z} \int_0^{\Delta x} \left[D_{yo} - \frac{\partial D_y}{\partial y}\frac{\Delta y}{2} \right] \boldsymbol{a}_y\cdot dx\,dz\,(-\boldsymbol{a}_y)$$

$$\iint_{\substack{top \\ face}} \boldsymbol{D}\cdot d\boldsymbol{s} + \iint_{\substack{bottom \\ face}} \boldsymbol{D}\cdot d\boldsymbol{s} = \int_0^{\Delta y} \int_0^{\Delta x} \left[D_{zo} + \frac{\partial D_z}{\partial z}\frac{\Delta z}{2} \right] \boldsymbol{a}_z\cdot dx\,dy\,\boldsymbol{a}_z$$

$$+ \int_0^{\Delta y} \int_0^{\Delta x} \left[D_{zo} - \frac{\partial D_z}{\partial z}\frac{\Delta z}{2} \right] \boldsymbol{a}_z\cdot dx\,dy\,(-\boldsymbol{a}_z)$$

$$\oint_{\Delta v} \boldsymbol{D} \cdot \boldsymbol{ds} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

The divergence operator in rectangular coordinates is then

$$\text{Div } \boldsymbol{D} = \nabla \cdot \boldsymbol{D} = \lim_{\Delta v \to 0} \left[\frac{\displaystyle\oint_{\Delta v} \boldsymbol{D} \cdot \boldsymbol{ds}}{\Delta v} \right] = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Note that the divergence operator can be expressed as the dot product of the gradient operator with the vector

$$\nabla \cdot \boldsymbol{D} = \left(\frac{\partial}{\partial x} \boldsymbol{a}_x + \frac{\partial}{\partial y} \boldsymbol{a}_y + \frac{\partial}{\partial z} \boldsymbol{a}_z \right) \cdot (D_x \boldsymbol{a}_x + D_y \boldsymbol{a}_y + D_z \boldsymbol{a}_z)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

The same process can be applied to the differential volume element in cylindrical and spherical coordinates. The results are shown below.

Cylindrical

$$\nabla \cdot \boldsymbol{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

Spherical

$$\nabla \cdot \boldsymbol{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (D_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial D_\phi}{\partial \phi}$$

<u>Example</u> (Divergence)

Given $\quad \boldsymbol{D} = \rho \sin\phi \, \boldsymbol{a}_\rho + \rho^2 z \, \boldsymbol{a}_\phi + z \cos\phi \, \boldsymbol{a}_z \quad$, determine ρ_V.

$$\rho_V = \nabla \cdot \boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$$

$$= \frac{1}{\rho}(2\rho\sin\phi) + \frac{1}{\rho}(0) + (\cos\phi)$$

$$= 2\sin\phi + \cos\phi \qquad (\text{C}/\text{m}^3)$$

Divergence Theorem

The *divergence theorem* (Gauss's theorem) is a vector theorem that allows a volume integral of the divergence of a vector to be transformed into a surface integral of the normal component of the vector, or vice verse. Given a volume V enclosed by a surface S and a vector $\boldsymbol{F}$ defined throughout V, the divergence theorem states

$$\oiint_S \boldsymbol{F} \cdot \boldsymbol{ds} = \iiint_V (\nabla \cdot \boldsymbol{F})\, dv \qquad (\text{Divergence theorem})$$

Gauss's law can be used to illustrate the validity of the divergence theorem.

$$\oiint_S \boldsymbol{D} \cdot \boldsymbol{ds} = Q_{enclosed} = \iiint_V \rho_V\, dv = \iiint_V (\nabla \cdot \boldsymbol{D})\, dv \qquad (\text{Gauss}'\text{s law})$$

<u>Example</u> (Divergence theorem and Gauss's law)

Using the divergence theorem, calculate the total charge within the volume V defined by $2 \le r \le 3,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le 2\pi$ given an electric flux density defined by $\boldsymbol{D} = r^2 \sin\phi\, \boldsymbol{a}_r + r\sin\theta\, \boldsymbol{a}_\theta$ (C/m^2) by evaluating

(a.) $Q_{enclosed} = \oiint_S \boldsymbol{D} \cdot \boldsymbol{ds}$

(b.) $Q_{enclosed} = \iiint_V (\nabla \cdot \boldsymbol{D})\, dv$

S_1 - outer hemispherical surface
$(r=3, \ 0 \le \theta \le \pi/2, \ 0 \le \phi \le 2\pi)$

S_2 - inner hemispherical surface
$(r=2, \ 0 \le \theta \le \pi/2, \ 0 \le \phi \le 2\pi)$

S_3 - flat ring
$(2 \le r \le 3, \ \theta = \pi/2, \ 0 \le \phi \le 2\pi)$

spherical coordinate differential volume
$$dv = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$$

(a.)
$$Q_{enclosed} = \oiint_S \boldsymbol{D} \cdot d\boldsymbol{s} = \iint_{S_1} \boldsymbol{D}_1 \cdot d\boldsymbol{s}_1 + \iint_{S_2} \boldsymbol{D}_2 \cdot d\boldsymbol{s}_2 + \iint_{S_3} \boldsymbol{D}_3 \cdot d\boldsymbol{s}_3$$

On $S_1 \ \Rightarrow \ \boldsymbol{D}_1 = \boldsymbol{D}(r=3) = 9\sin\phi\,\boldsymbol{a}_r + 3\sin\theta\,\boldsymbol{a}_\theta$ $d\boldsymbol{s}_1 = 9\sin\theta\,d\theta\,d\phi\,(\boldsymbol{a}_r)$

On $S_2 \ \Rightarrow \ \boldsymbol{D}_2 = \boldsymbol{D}(r=2) = 4\sin\phi\,\boldsymbol{a}_r + 2\sin\theta\,\boldsymbol{a}_\theta$ $d\boldsymbol{s}_2 = 4\sin\theta\,d\theta\,d\phi\,(-\boldsymbol{a}_r)$

On $S_3 \ \Rightarrow \ \boldsymbol{D}_3 = \boldsymbol{D}(\theta=\pi/2) = r^2\sin\phi\,\boldsymbol{a}_r + r\,\boldsymbol{a}_\theta$ $d\boldsymbol{s}_3 = r\,dr\,d\phi\,(\boldsymbol{a}_\theta)$

$$Q_{enclosed} = 81 \int_0^{2\pi} \int_0^{\pi/2} \overset{0}{\cancel{\sin\phi}}\sin\theta\,d\theta\,d\phi - 16 \int_0^{2\pi} \int_0^{\pi/2} \overset{0}{\cancel{\sin\phi}}\sin\theta\,d\theta\,d\phi$$

$$+ \int_0^{2\pi} \int_2^{3} r^2\,dr\,d\phi$$

$$Q_{enclosed} = \left[\frac{r^3}{3}\right]_2^3 [\phi]_0^{2\pi} = \frac{27-8}{3}(2\pi) = \frac{38}{3}\pi \ (C)$$

(b.)

$$Q_{enclosed} = \oiiint_V (\nabla \cdot D)\, dv$$

$$\nabla \cdot D = \rho_V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(D_\theta \sin\theta)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4 \sin\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(r\sin^2\theta)$$

$$= \frac{1}{r^2}(4r^3 \sin\phi) + \frac{1}{r\sin\theta}(2r\sin\theta\cos\theta)$$

$$= 4r\sin\phi + 2\cos\theta \quad (C/m^3)$$

$$Q_{enclosed} = \int_0^{2\pi}\int_0^{\pi/2}\int_2^3 (4r\sin\phi + 2\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 4\int_0^{2\pi}\int_0^{\pi/2}\int_2^3 (r^3\sin\theta\sin\phi)\, dr\, d\theta\, d\phi \nearrow^{0}$$

$$+ 2\int_0^{2\pi}\int_0^{\pi/2}\int_2^3 (r^2\sin\theta\cos\theta)\, dr\, d\theta\, d\phi$$

$$\int \sin\theta\cos\theta\, d\theta = \frac{1}{2}\sin^2\theta$$

$$Q_{enclosed} = 2\left[\frac{r^3}{3}\right]_2^3 \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} [\phi]_0^{2\pi}$$

$$= 2\left(\frac{19}{3}\right)\left(\frac{1}{2}\right)(2\pi) = \frac{38}{3}\pi \ (C)$$

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Electric Scalar Potential

Given that the electric field defines the force per unit charge acting on a positive test charge, any attempt to move the test charge against the electric field requires that work be performed. The *potential difference* between two points in an electric field is defined as the work per unit charge performed when moving a positive test charge from one point to the other.

From Coulomb's law, the vector force on a positive point charge in an electric field is given by

$$\boldsymbol{F} = Q\boldsymbol{E}$$

The amount of work performed in moving this point charge in the electric field is product of the force and the distance moved. When the positive point charge is moved against the force (against the electric field), the work done is positive. When the point charge is moved in the direction of the force, the work done is negative. If the point charge is moved in a direction perpendicular to the force, the amount of work done is zero. For a differential element of length (*dl*), the small amount of work done (*dW*) is defined as

$$dW = -\boldsymbol{F} \cdot d\boldsymbol{l} = -Q\boldsymbol{E} \cdot d\boldsymbol{l}$$

The minus sign in the previous equation is necessary to obtain the proper sign on the work done (positive when moving the test charge against the electric field). When the point charge is moved along a path from point A to B, the total amount of work performed (W) is found by integrating dW along the path.

$$W = \int_A^B dW = -\int_A^B \boldsymbol{F} \cdot d\boldsymbol{l} = -Q \int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} \qquad (\text{J})$$

The potential difference between A and B is then

$$V_{AB} = \frac{W}{Q} = -\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} \qquad \left(\frac{\text{J}}{\text{C}} = \text{V} \right)$$

The potential difference equation may be written as

$$V_{AB} = -\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} = \int_A^B dV = V_B - V_A$$

where V_A and V_B are the *absolute potentials* at points A and B, respectively. The absolute potential at a point is defined as the potential difference between the point and a reference point an infinite distance away. The definition of the potential difference in terms of the absolute potentials at the starting and ending points of the path shows that the potential difference between any two points is independent of the path taken between the points.

For a closed path (point A = point B), the line integral of the electric field yields the potential difference between a point and itself yielding a value of zero.

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

Vector fields which have zero-valued closed path line integrals are designated as *conservative* fields. All electrostatic fields are conservative fields.
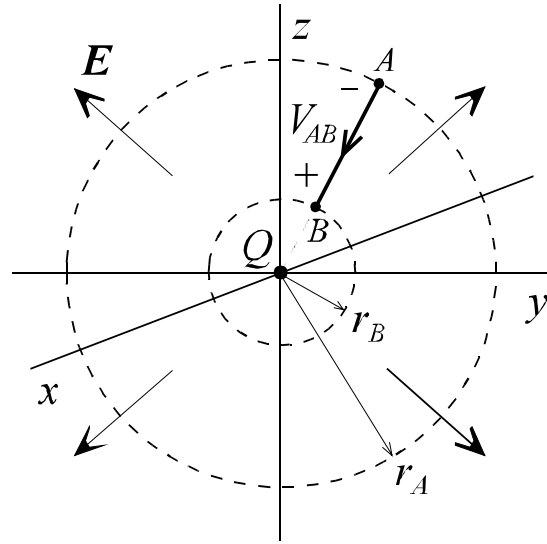
<u>Example</u> (Potential difference)

Determine the absolute potential in the electric field of a point charge Q located at the coordinate origin.

The electric field of a point charge at the origin is

$$E = \frac{Q}{4\pi\epsilon_o r^2} \, a_r$$

The potential difference between two points A and B in the electric field of the point charge is

$$V_{AB} = -\int_A^B E \cdot dl = \int_A^B dV = V_B - V_A$$

If we choose an inward radial path from $r = r_A$ to $r = r_B$, the vector differential length is

$$dl = dl\,a_l = (-dr)(-a_r) = dr\,a_r$$

which yields

$$V_{AB} = -\int_{r_A}^{r_B} \left(\frac{Q}{4\pi\epsilon_o r^2} \, a_r \right) \cdot (dr\,a_r) = -\frac{Q}{4\pi\epsilon_o} \int_{r_A}^{r_B} \frac{dr}{r^2}$$

$$= -\frac{Q}{4\pi\epsilon_o} \left[-\frac{1}{r} \right]_{r_A}^{r_B} = \frac{Q}{4\pi\epsilon_o} \left[\frac{1}{r_B} - \frac{1}{r_A} \right]$$

The absolute potential at point B is found by taking the limit as r_A approaches infinity.

$$V_B = \lim_{r_A \to \infty} V_{AB} = \frac{Q}{4\pi\epsilon_o} \lim_{r_A \to \infty} \left[\frac{1}{r_B} - \frac{1}{r_A} \right] = \frac{Q}{4\pi\epsilon_o r_B}$$
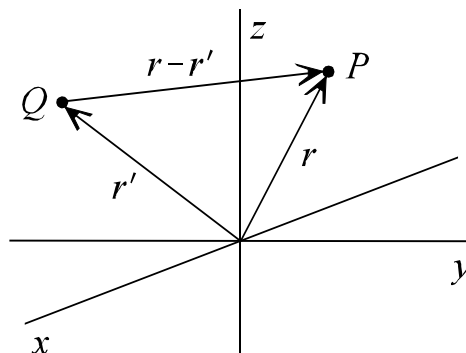
Potentials of Charge Distributions

The previous formula can be generalized as the absolute potential of a point charge at the origin (let $r_B = r$).

$$V = \frac{Q}{4\pi\epsilon_o r}$$

(Absolute potential for a point charge at the origin)

Note that the potential distribution of the point charge exhibits spherical symmetry just like the electric field. The potential of the point charge varies as r^{-1} in comparison to the electric field of a point charge which varies as r^{-2}. Surfaces on which the potential is constant are designated as *equipotential surfaces*. Equipotential surfaces are always perpendicular to the electric field (since no work is performed to move a charge perpendicular to the electric field). For the point charge, the equipotential surfaces are concentric spherical surfaces about the point charge.

The absolute potential of a point charge at an arbitrary location is

$$V = \frac{Q}{4\pi\epsilon_o |\mathbf{r} - \mathbf{r}'|}$$

(Absolute potential for a point charge at an arbitrary location)

The principle of superposition can be applied to the determine the potential due to a set of point charges which yields

$$V = \frac{1}{4\pi\epsilon_o} \sum_{k=1}^{N} \frac{Q_k}{|\mathbf{r} - \mathbf{r}'_k|}$$

(Absolute potential of a set of point charges)

The potentials due to line, surface and volume distributions of charge are found by integrating the incremental potential contribution due to each differential element of charge in the distribution.

Point Charge

$$V(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_o R}$$

$$R = |\boldsymbol{r} - \boldsymbol{r}'|$$

Line Charge $(\rho_L\, dl' \leftrightarrow Q)$

$$dV(\boldsymbol{r}) = \frac{\rho_L\, dl'}{4\pi\epsilon_o R}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_o} \int_L \frac{\rho_L}{R}\, dl'$$

Surface Charge $(\rho_S\, ds' \leftrightarrow Q)$

$$dV(\boldsymbol{r}) = \frac{\rho_S\, ds'}{4\pi\epsilon_o R}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_o} \iint_S \frac{\rho_S}{R}\, ds'$$

Volume Charge $(\rho_V\, dv' \leftrightarrow Q)$

$$dV(\boldsymbol{r}) = \frac{\rho_V\, dv'}{4\pi\epsilon_o R}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_o} \iiint_V \frac{\rho_V}{R}\, dv'$$

<u>Example</u> (Potential due to a line charge)

Determine the potential in the *x-y* plane due to a uniform line charge of length $2a$ lying along the *z*-axis and centered at the coordinate origin

$$V = \frac{1}{4\pi\epsilon_o} \int_L \frac{\rho_L}{R} \, dl'$$

$$dl' = dz'$$

$$\mathbf{r} = \rho \, \mathbf{a}_\rho$$

$$\mathbf{r}' = z' \, \mathbf{a}_z$$

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \rho \, \mathbf{a}_\rho - z' \mathbf{a}_z$$

$$R = |\mathbf{R}| = \sqrt{z'^2 + \rho^2}$$

$$V = \frac{\rho_L}{4\pi\epsilon_o} \int_{-a}^{a} \frac{dz'}{\sqrt{z'^2 + \rho^2}} = (2) \frac{\rho_L}{4\pi\epsilon_o} \int_{0}^{a} \frac{dz'}{\sqrt{z'^2 + \rho^2}}$$

Even integrand
Symmetric limits

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln[x + \sqrt{x^2 + a^2}]$$

$$V = \frac{\rho_L}{2\pi\epsilon_o} \ln[z' + \sqrt{z'^2 + \rho^2}] \Big]_0^a = \frac{\rho_L}{2\pi\epsilon_o} \left\{ \ln[a + \sqrt{a^2 + \rho^2}] - \ln\rho \right\}$$

$$= \frac{\rho_L}{2\pi\epsilon_o} \ln\left[\frac{a + \sqrt{\rho^2 + a^2}}{\rho} \right]$$

(Absolute potential in the *x-y* plane due to a uniform line charge of length $2a$ lying along the *z*-axis centered at the coordinate origin)

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<u>Example</u> (Potential due to a square loop)

Determine the potential at the center of a square loop of side length l which is uniformly charged.

The uniformly charged square loop can be viewed as four line charges. The total potential at the center of the loop is the scalar sum of the contributions from the four sides   (identical scalar contributions). Thus, the potential at P due to one side of the loop is

$$V_{side} = \frac{\rho_L}{2 \pi \epsilon_o} \ln \left[\frac{a + \sqrt{\rho^2 + a^2}}{\rho} \right] \qquad a = \frac{l}{2} \qquad \rho = \frac{l}{2}$$

$$V_{total} = 4 V_{side} = (4) \frac{\rho_L}{2 \pi \epsilon_o} \ln \left[\frac{l/2 + \sqrt{(l/2)^2 + (l/2)^2}}{l/2} \right]$$

$$= \frac{2 \rho_L}{\pi \epsilon_o} \ln [1 + \sqrt{2}]$$

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Electric Field as the Gradient of the Potential

The potential difference between two points in an electric field can be written as the line integral of the electric field such that

$$V_{AB} = -\int_{A}^{B} \boldsymbol{E} \cdot d\boldsymbol{l} = \int_{A}^{B} dV = V_B - V_A$$

From the equation above, the incremental change in potential along the integral path is

$$dV = -\boldsymbol{E} \cdot d\boldsymbol{l} = -\boldsymbol{E} \cdot \boldsymbol{a}_l \, dl = -E\cos\theta \, dl$$

where θ is the angle between the direction of the integral path and the electric field. The derivative of the potential with respect to position along the path may be written as

$$\frac{dV}{dl} = -E\cos\theta$$

Note that the potential derivative is a maximum when $\theta = \pi$ (when the direction of the electric field is opposite to the direction of the path). Thus,

$$\left[\frac{dV}{dl}\right]_{\text{max}} = E \qquad \text{when } \theta = \pi \quad (\cos\theta = -1)$$

This equation shows that the magnitude of the electric field is equal to the maximum space rate of change in the potential. The direction of the electric field is the direction of the maximum decrease in the potential (the electric field always points from a region of higher potential to a region of lower potential).

The electric field can be written in terms of the potential as

$$\boldsymbol{E} = \frac{dV}{dl}(-\boldsymbol{a}_l) = -\frac{dV}{dl}\boldsymbol{a}_l = -\nabla V$$

where the operator "∇" (del) is the *gradient operator*. The gradient operator is a differential operator which operates on a scalar function to yield (1) the maximum increase per unit distance and (2) the direction of the maximum increase. Since the electric field always points in the direction of decreasing potential, the electric field is the negative of the gradient of *V*.

The derivative with respect to *l* in the gradient operator above can be generalized to a particular coordinate system by including the variation in the potential with respect to the three coordinate variables. In rectangular coordinates,

$$\nabla V = \frac{dV}{dl}\boldsymbol{a}_l = \frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial y}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z$$

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{a}_x + \frac{\partial}{\partial y}\boldsymbol{a}_y + \frac{\partial}{\partial z}\boldsymbol{a}_z \qquad \text{(Gradient operator in rectangular coordinates)}$$

The gradient operator is defined differently in rectangular, cylindrical and spherical coordinates. The electric field expression as the gradient of the potential in these coordinate systems are

$$\boldsymbol{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial y}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right] \qquad (\text{rectangular})$$

$$= -\left[\frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right] \qquad (\text{cylindrical})$$

$$= -\left[\frac{\partial V}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi\right] \qquad (\text{spherical})$$

<u>Example</u> (E as the gradient of V)

Given $V(r,\theta,\phi) = \dfrac{10}{r^2}\sin\theta\cos\phi$ (a.) find $E(r,\theta,\phi)$ and (b.) E at $(2,\pi/2,0)$.

(a.)

$$E = -\nabla V = -\left[\frac{\partial V}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\boldsymbol{a}_\phi\right]$$

$$= -\left[10\sin\theta\cos\phi\,\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\boldsymbol{a}_r + \frac{10}{r^3}\cos\phi\,\frac{\partial}{\partial\theta}(\sin\theta)\boldsymbol{a}_\theta\right.$$

$$\left.+\frac{10}{r^3}\frac{\partial}{\partial\phi}(\cos\phi)\,\boldsymbol{a}_\phi\right]$$

$$= \frac{20}{r^3}\sin\theta\cos\phi\,\boldsymbol{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\boldsymbol{a}_\theta + \frac{10}{r^3}\sin\phi\,\boldsymbol{a}_\phi$$

$$= E_r\boldsymbol{a}_r + E_\theta\boldsymbol{a}_\theta + E_\phi\boldsymbol{a}_\phi$$

(b.)

$$E(2,\pi/2,0) = \frac{20}{8}\boldsymbol{a}_r + 0\,\boldsymbol{a}_\theta + 0\,\boldsymbol{a}_\phi = \frac{20}{8}\boldsymbol{a}_r\ (\text{V/m})$$

Summary of Electric Field / Potential Relationships

$V = -\displaystyle\int E\cdot d\boldsymbol{l}$ Integrate E to find V

$E = -\nabla V$ Differentiate V to find E

Electric Flux Density

The *electric flux density* D in free space is defined as the product of the free space permittivity (ϵ_o) and the electric field (E):

$$D = \epsilon_o E$$

Given that the electric field is inversely proportional to the permittivity of the medium, the electric flux density is independent of the medium properties.

point charge	$E = \dfrac{Q}{4\pi\epsilon_o r^2} a_r$	$D = \dfrac{Q}{4\pi r^2} a_r$
line charge	$E = \dfrac{\rho_L}{2\pi\epsilon_o \rho} a_\rho$	$D = \dfrac{\rho_L}{2\pi\rho} a_\rho$

The units on electric flux density are

$$\frac{F}{m} \times \frac{V}{m} = \frac{C}{m^2}$$

so that the units on electric flux density are equivalent to surface charge density.

The *total electric flux* (ψ) passing through a surface S is defined as the integral of the normal component of D through the surface.

$$\psi = \iint_S D \cdot ds = \iint_S D \cdot a_n \, ds = \iint_S D_n \, ds$$

where a_n is the unit normal to the surface S and D_n is the component of D normal to S. The direction chosen for the unit normal (one of two possible) defines the direction of the total flux.

For a closed surface, the total electric flux is

$$\psi = \oint_S D \cdot ds$$

$a_n -$ outward (total outward flux)
$a_n -$ inward (total inward flux)

Gauss's Law

Gauss's law is one of the set of four *Maxwell's equations* that govern the behavior of electromagnetic fields.

Gauss's Law - *The total outward electric flux ψ through any closed surface is equal to the total charge enclosed by the surface.*

Gauss's law is written in equation form as

$$\psi = \oint_S \boldsymbol{D} \cdot \boldsymbol{ds} = Q_{enclosed} \qquad (\text{Gauss's law})$$

where $ds = \boldsymbol{a}_n\, ds$ and $\boldsymbol{a}_n$ is the *outward* pointing unit normal to S.

<u>Example</u> (Gauss's law, point charge at origin)
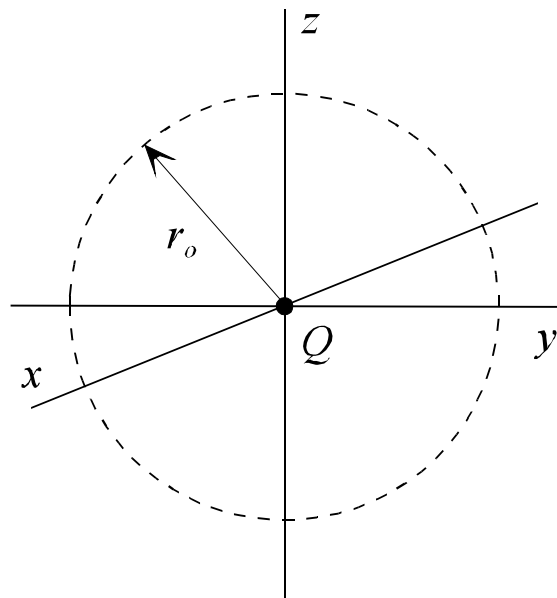
Given a point charge at the origin, show that Gauss's law is valid on a spherical surface (S) of radius r_o.

Gauss's law applied to the spherical surface S surrounding the point charge Q at the origin should yield

$$\psi = \oint_S \boldsymbol{D} \cdot \boldsymbol{ds} = Q_{enclosed} = Q$$

The electric flux produced by Q is

$$\boldsymbol{D} = \frac{Q}{4\pi r^2}\, \boldsymbol{a}_r$$

On the spherical surface S of radius r_o, we have

$$\boldsymbol{D}(r=r_o) = \frac{Q}{4\pi r_o^2}\,\boldsymbol{a}_r \qquad\qquad ds = r_o^2 \sin\theta\, d\theta\, d\phi\,\boldsymbol{a}_r$$

$$\boldsymbol{D}\cdot ds = \left(\frac{Q}{4\pi r_o^2}\,\boldsymbol{a}_r\right)\cdot\left(r_o^2 \sin\theta\, d\theta\, d\phi\,\boldsymbol{a}_r\right) = \frac{Q}{4\pi}\sin\theta\, d\theta\, d\phi$$

$$\psi = \oint_S \boldsymbol{D}\cdot ds = \frac{Q}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\sin\theta\, d\theta\, d\phi = \frac{Q}{4\pi}\Big[-\cos\theta\Big]_0^{\pi}\Big[\phi\Big]_0^{2\pi}$$

$$= \frac{Q}{4\pi}(2)(2\pi) = Q \qquad (\text{charge enclosed})$$

Note the **outward** pointing normal requirement in Gauss's law is a direct result of our electric field (flux) convention.

interior positive charges $\rightarrow$ **outward electric flux**

interior negative charges $\rightarrow$ **inward electric flux**

By using an outward pointing normal, we obtain the correct sign on the enclosed charge.

Gauss's law can also be used to determine the electric fields produced by simple charge distributions that exhibit special symmetry. Examples of such charge distributions include uniformly charged spherical surfaces and volumes.

<u>Example</u> (Using Gauss's law to determine E)

Use Gauss's law to determine the vector electric field inside and outside a uniformly charged spherical volume of radius a.

$$\rho_V = \begin{cases} k & r<a \\ 0 & r>a \end{cases}$$

$k = $ constant

S – spherical surface of radius $r = a$

S_+– spherical surface of radius $r > a$

S_-– spherical surface of radius $r < a$

Gauss's law can be applied on S_- to determine the electric field inside the charged sphere [$E(r<a)$].

$$\oint_{S_-} D \cdot ds = Q_{enclosed} = \oint_{V_-} \rho_V \, dv$$

$(V_- = \text{volume enclosed by } S_-)$

$$ds = ds \, a_r$$

$$\oint_{S_-} D \cdot ds = \oint_{S_-} D \cdot a_r \, ds$$

$$= \oint_{S_-} D_r \, ds = Q_{enclosed}$$

By symmetry, on S_- (and S_+), D_r is uniform and has only an a_r component.

$$\oint_{S_-} D_r\, ds = D_r \oint_{S_-} ds = D_r\, (4\pi r^2) = Q_{enclosed}$$

$$= \oint_{V_-} \rho_V\, dv = \rho_V \oint_{V_-} dv = k\left(\frac{4}{3}\pi r^3\right)$$

$$D_r = \frac{k\left(\dfrac{4}{3}\pi r^3\right)}{4\pi r^2} = \frac{kr}{3} \qquad (r<a)$$

or

$$D = \frac{kr}{3}\, a_r \qquad E = \frac{D}{\epsilon_o} = \frac{kr}{3\epsilon_o}\, a_r \qquad (r<a)$$

Gauss's law can be applied on S_+ to determine the electric field outside the charged sphere [$E(r>a)$].

$$\oint_{S_+} D_r\, ds = D_r \oint_{S_+} ds = D_r\, (4\pi r^2) = Q_{enclosed}$$

$$= \oint_{V_+} \rho_V\, dv = \rho_V \oint_{V} dv = k\left(\frac{4}{3}\pi a^3\right)$$

$$(V = \text{volume enclosed by } S)$$

$$D_r = \frac{k\left(\dfrac{4}{3}\pi a^3\right)}{4\pi r^2} = \frac{ka^3}{3r^2} \qquad (r>a)$$

or

$$D = \frac{ka^3}{3r^2}\, a_r \qquad E = \frac{ka^3}{3\epsilon_o r^2}\, a_r \qquad (r>a)$$

Electric Field for the uniformly charged spherical volume of radius a

$$\frac{kr}{3\epsilon_o}$$

$$|E|$$

$$\frac{ka}{3\epsilon_o}$$

$$\frac{ka^3}{3\epsilon_o r^2}$$

a

r

Divergence Operator / Gauss's Law (Differential Form)

The differential form of Gauss's law is determined by applying the integral form of Gauss's law to a differential volume (Δv). The differential form of Gauss's law is defined in terms of the *divergence operator*. The divergence operator is obtained by taking the limit as Δv shrinks to zero (to the point P) of the flux out of Δv divided by Δv.

$$\Delta v = \Delta x \Delta y \Delta z$$

$$P = (x_o, y_o, z_o)$$

$$\oint_{\Delta v} \boldsymbol{D} \cdot \boldsymbol{ds} \qquad (\text{net flux out of } \Delta v)$$

$$\text{Div } \boldsymbol{D} = \nabla \cdot \boldsymbol{D} \equiv \lim_{\Delta v \to 0} \left[\frac{\oint_{\Delta v} \boldsymbol{D} \cdot \boldsymbol{ds}}{\Delta v} \right] = \lim_{\Delta v \to 0} \left[\frac{Q_{enclosed}}{\Delta v} \right] = \rho_V(P)$$

Gradient operator

$$\oint_S \boldsymbol{D} \cdot \boldsymbol{ds} = Q_{enclosed} \qquad \text{Gauss's law (integral form)}$$

$$\nabla \cdot \boldsymbol{D} = \rho_V \qquad \text{Gauss's law (differential form)}$$

The divergence operator in rectangular coordinates can be determined by performing the required integrations. The electric flux density within the differential volume is defined by

$$\boldsymbol{D} = D_x \boldsymbol{a}_x + D_y \boldsymbol{a}_y + D_z \boldsymbol{a}_z$$

while the electric flux density evaluated at the point P is defined as

$$\boldsymbol{D}(P) = D_{xo} \boldsymbol{a}_x + D_{yo} \boldsymbol{a}_y + D_{zo} \boldsymbol{a}_z$$

The total flux out of the differential volume Δv is

$$\oint_{\Delta v} \boldsymbol{D} \cdot d\boldsymbol{s} = \iint_{\substack{front \\ face}} \boldsymbol{D} \cdot d\boldsymbol{s} + \iint_{\substack{back \\ face}} \boldsymbol{D} \cdot d\boldsymbol{s} + \iint_{\substack{left \\ face}} \boldsymbol{D} \cdot d\boldsymbol{s} + \iint_{\substack{right \\ face}} \boldsymbol{D} \cdot d\boldsymbol{s} + \iint_{\substack{top \\ face}} \boldsymbol{D} \cdot d\boldsymbol{s} + \iint_{\substack{bottom \\ face}} \boldsymbol{D} \cdot d\boldsymbol{s}$$

The electric flux density components can be written in terms of a Taylor series about the point P.

$$D_x(x) = D_{xo} + \frac{\partial D_x}{\partial x}(x - x_o) + \frac{\partial^2 D_x}{\partial x^2}\frac{(x - x_o)^2}{2!} + \ldots$$

$$D_y(y) = D_{yo} + \frac{\partial D_y}{\partial y}(y - y_o) + \frac{\partial^2 D_y}{\partial y^2}\frac{(y - y_o)^2}{2!} + \ldots$$

$$D_z(z) = D_{zo} + \frac{\partial D_z}{\partial z}(z - z_o) + \frac{\partial^2 D_z}{\partial z^2}\frac{(z - z_o)^2}{2!} + \ldots$$

For points close to P (such as the faces on the differential volume), the higher order terms in the Taylor series expansions become negligible such that

$$D_x(x) \approx D_{xo} + \frac{\partial D_x}{\partial x}(x - x_o)$$

$$D_y(y) \approx D_{yo} + \frac{\partial D_y}{\partial y}(y - y_o)$$

$$D_z(z) \approx D_{zo} + \frac{\partial D_z}{\partial z}(z - z_o)$$

The flux densities on the six faces of the differential volume are

<u>front face</u>

$$D_x(x_o + \Delta x/2) \approx D_{xo} + \frac{\partial D_x}{\partial x}\frac{\Delta x}{2}$$

<u>back face</u>

$$D_x(x_o - \Delta x/2) \approx D_{xo} - \frac{\partial D_x}{\partial x}\frac{\Delta x}{2}$$

<u>right face</u>

$$D_y(y_o + \Delta y/2) \approx D_{yo} + \frac{\partial D_y}{\partial y}\frac{\Delta y}{2}$$

<u>left face</u>

$$D_y(y_o - \Delta y/2) \approx D_{yo} - \frac{\partial D_y}{\partial y}\frac{\Delta y}{2}$$

<u>top face</u>

$$D_z(z_o + \Delta z/2) \approx D_{zo} + \frac{\partial D_z}{\partial z}\frac{\Delta z}{2}$$

<u>bottom face</u>

$$D_z(z_o - \Delta z/2) \approx D_{zo} - \frac{\partial D_z}{\partial z}\frac{\Delta z}{2}$$

The integrations over the six sides of the differential volume yield

$$\iint_{\substack{front\\face}} \boldsymbol{D}\cdot d\boldsymbol{s} + \iint_{\substack{back\\face}} \boldsymbol{D}\cdot d\boldsymbol{s} = \int_0^{\Delta z}\int_0^{\Delta y}\left[D_{xo} + \frac{\partial D_x}{\partial x}\frac{\Delta x}{2}\right]\boldsymbol{a}_x\cdot dy\,dz\,\boldsymbol{a}_x$$

$$+ \int_0^{\Delta z}\int_0^{\Delta y}\left[D_{xo} - \frac{\partial D_x}{\partial x}\frac{\Delta x}{2}\right]\boldsymbol{a}_x\cdot dy\,dz\,(-\boldsymbol{a}_x)$$

$$\iint_{\substack{right\\face}} \boldsymbol{D}\cdot d\boldsymbol{s} + \iint_{\substack{left\\face}} \boldsymbol{D}\cdot d\boldsymbol{s} = \int_0^{\Delta z}\int_0^{\Delta x}\left[D_{yo} + \frac{\partial D_y}{\partial y}\frac{\Delta y}{2}\right]\boldsymbol{a}_y\cdot dx\,dz\,\boldsymbol{a}_y$$

$$+ \int_0^{\Delta z}\int_0^{\Delta x}\left[D_{yo} - \frac{\partial D_y}{\partial y}\frac{\Delta y}{2}\right]\boldsymbol{a}_y\cdot dx\,dz\,(-\boldsymbol{a}_y)$$

$$\iint_{\substack{top\\face}} \boldsymbol{D}\cdot d\boldsymbol{s} + \iint_{\substack{bottom\\face}} \boldsymbol{D}\cdot d\boldsymbol{s} = \int_0^{\Delta y}\int_0^{\Delta x}\left[D_{zo} + \frac{\partial D_z}{\partial z}\frac{\Delta z}{2}\right]\boldsymbol{a}_z\cdot dx\,dy\,\boldsymbol{a}_z$$

$$+ \int_0^{\Delta y}\int_0^{\Delta x}\left[D_{zo} - \frac{\partial D_z}{\partial z}\frac{\Delta z}{2}\right]\boldsymbol{a}_z\cdot dx\,dy\,(-\boldsymbol{a}_z)$$

$$\oint_{\Delta v} \boldsymbol{D} \cdot d\boldsymbol{s} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

The divergence operator in rectangular coordinates is then

$$\text{Div } \boldsymbol{D} = \nabla \cdot \boldsymbol{D} = \lim_{\Delta v \to 0} \left[\frac{\oint_{\Delta v} \boldsymbol{D} \cdot d\boldsymbol{s}}{\Delta v} \right] = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Note that the divergence operator can be expressed as the dot product of the gradient operator with the vector

$$\nabla \cdot \boldsymbol{D} = \left(\frac{\partial}{\partial x} \boldsymbol{a}_x + \frac{\partial}{\partial y} \boldsymbol{a}_y + \frac{\partial}{\partial z} \boldsymbol{a}_z \right) \cdot (D_x \boldsymbol{a}_x + D_y \boldsymbol{a}_y + D_z \boldsymbol{a}_z)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

The same process can be applied to the differential volume element in cylindrical and spherical coordinates. The results are shown below.

Cylindrical

$$\nabla \cdot \boldsymbol{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

Spherical

$$\nabla \cdot \boldsymbol{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (D_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial D_\phi}{\partial \phi}$$

<u>Example</u> (Divergence)
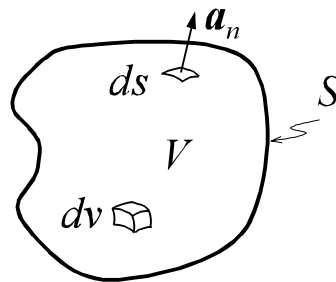
Given $\boldsymbol{D} = \rho\sin\phi\,\boldsymbol{a}_\rho + \rho^2 z\,\boldsymbol{a}_\phi + z\cos\phi\,\boldsymbol{a}_z$, determine ρ_V.

$$\rho_V = \nabla\cdot\boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$$

$$= \frac{1}{\rho}(2\rho\sin\phi) + \frac{1}{\rho}(0) + (\cos\phi)$$

$$= 2\sin\phi + \cos\phi \qquad (C/m^3)$$

Divergence Theorem

The *divergence theorem* (Gauss's theorem) is a vector theorem that allows a volume integral of the divergence of a vector to be transformed into a surface integral of the normal component of the vector, or vice verse. Given a volume V enclosed by a surface S and a vector $\boldsymbol{F}$ defined throughout V, the divergence theorem states

$$\oiint_S \boldsymbol{F} \cdot \boldsymbol{ds} = \iiint_V (\nabla \cdot \boldsymbol{F}) \, dv \qquad (\text{Divergence theorem})$$

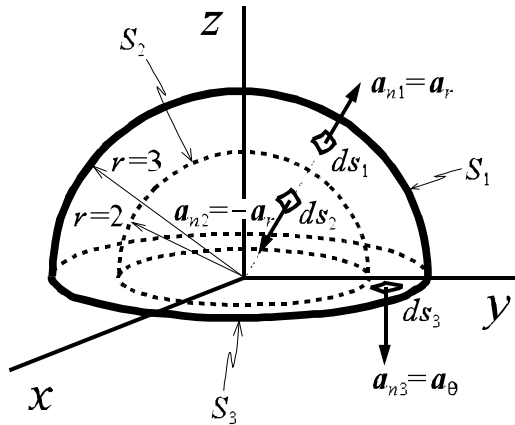Gauss's law can be used to illustrate the validity of the divergence theorem.

$$\oiint_S \boldsymbol{D} \cdot \boldsymbol{ds} = Q_{enclosed} = \iiint_V \rho_V \, dv = \iiint_V (\nabla \cdot \boldsymbol{D}) \, dv \qquad (\text{Gauss}'\text{s law})$$

<u>Example</u> (Divergence theorem and Gauss's law)

Using the divergence theorem, calculate the total charge within the volume V defined by $2 \le r \le 3$, $0 \le \theta \le \pi/2$, $0 \le \phi \le 2\pi$ given an electric flux density defined by $\boldsymbol{D} = r^2 \sin\phi \, \boldsymbol{a}_r + r\sin\theta \, \boldsymbol{a}_\theta$ (C/m^2)   by evaluating

(a.) $Q_{enclosed} = \oiint_S \boldsymbol{D} \cdot \boldsymbol{ds}$

(b.) $Q_{enclosed} = \iiint_V (\nabla \cdot \boldsymbol{D}) \, dv$

S_1 - outer hemispherical surface
$$(r=3, \ 0 \le \theta \le \pi/2, \ 0 \le \phi \le 2\pi)$$

S_2 - inner hemispherical surface
$$(r=2, \ 0 \le \theta \le \pi/2, \ 0 \le \phi \le 2\pi)$$

S_3 - flat ring
$$(2 \le r \le 3, \ \theta = \pi/2, \ 0 \le \phi \le 2\pi)$$

spherical coordinate differential volume
$$dv = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$$

(a.) $$Q_{enclosed} = \oiint_{S} \boldsymbol{D} \cdot d\boldsymbol{s} = \iint_{S_1} \boldsymbol{D}_1 \cdot d\boldsymbol{s}_1 + \iint_{S_2} \boldsymbol{D}_2 \cdot d\boldsymbol{s}_2 + \iint_{S_3} \boldsymbol{D}_3 \cdot d\boldsymbol{s}_3$$

On $S_1 \Rightarrow \boldsymbol{D}_1 = \boldsymbol{D}(r=3) = 9\sin\phi\,\boldsymbol{a}_r + 3\sin\theta\,\boldsymbol{a}_\theta \quad d\boldsymbol{s}_1 = 9\sin\theta\,d\theta\,d\phi(\boldsymbol{a}_r)$

On $S_2 \Rightarrow \boldsymbol{D}_2 = \boldsymbol{D}(r=2) = 4\sin\phi\,\boldsymbol{a}_r + 2\sin\theta\,\boldsymbol{a}_\theta \quad d\boldsymbol{s}_2 = 4\sin\theta\,d\theta\,d\phi(-\boldsymbol{a}_r)$

On $S_3 \Rightarrow \boldsymbol{D}_3 = \boldsymbol{D}(\theta = \pi/2) = r^2\sin\phi\,\boldsymbol{a}_r + r\,\boldsymbol{a}_\theta \quad d\boldsymbol{s}_3 = r\,dr\,d\phi(\boldsymbol{a}_\theta)$

$$Q_{enclosed} = 81 \int_0^{2\pi} \int_0^{\pi/2} \sin\phi\sin\theta\,d\theta\,d\phi - 16 \int_0^{2\pi} \int_0^{\pi/2} \sin\phi\sin\theta\,d\theta\,d\phi$$

$$+ \int_0^{2\pi} \int_2^3 r^2\,dr\,d\phi$$

$$Q_{enclosed} = \left[\frac{r^3}{3}\right]_2^3 [\phi]_0^{2\pi} = \frac{27-8}{3}(2\pi) = \frac{38}{3}\pi \ (C)$$

(b.)

$$Q_{enclosed} = \oiiint_V (\nabla \cdot \boldsymbol{D})\, dv$$

$$\nabla \cdot \boldsymbol{D} = \rho_V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4 \sin\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta)$$

$$= \frac{1}{r^2}(4r^3\sin\phi) + \frac{1}{r\sin\theta}(2r\sin\theta\cos\theta)$$

$$= 4r\sin\phi + 2\cos\theta \quad (C/m^3)$$

$$Q_{enclosed} = \int_0^{2\pi}\int_0^{\pi/2}\int_2^{3} (4r\sin\phi + 2\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 4\int_0^{2\pi}\int_0^{\pi/2}\int_2^{3} (r^3\sin\theta\sin\phi)\, dr\, d\theta\, d\phi \nearrow^{0}$$

$$+ 2\int_0^{2\pi}\int_0^{\pi/2}\int_2^{3} (r^2\sin\theta\cos\theta)\, dr\, d\theta\, d\phi$$

$$\int \sin\theta\cos\theta\, d\theta = \frac{1}{2}\sin^2\theta$$

$$Q_{enclosed} = 2\left[\frac{r^3}{3}\right]_2^{3}\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}[\phi]_0^{2\pi}$$

$$= 2\left(\frac{19}{3}\right)\left(\frac{1}{2}\right)(2\pi) = \frac{38}{3}\pi \ (C)$$
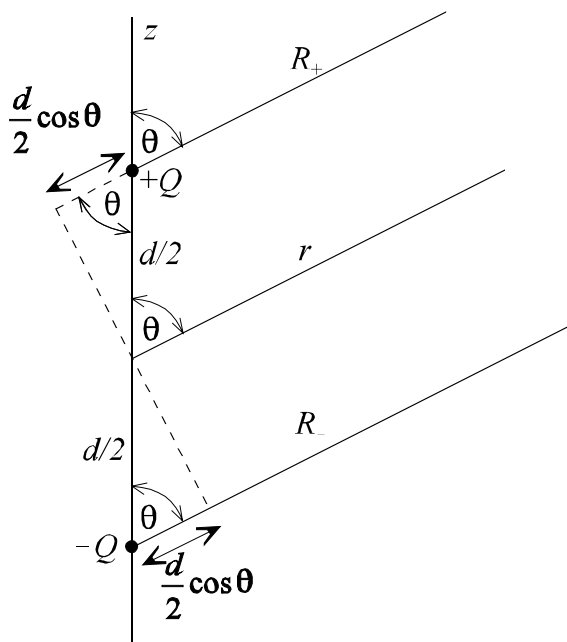
Electric Dipole

An *electric dipole* is formed by two point charges of equal magnitude and opposite sign ($+Q$, $-Q$) separated by a short distance d. The potential at the point P due to the electric dipole is found using superposition.

$$V = V_+ + V_-$$

$$= \frac{Q}{4\pi\epsilon_o R_+} + \frac{-Q}{4\pi\epsilon_o R_-}$$

$$= \frac{Q}{4\pi\epsilon_o}\left[\frac{1}{R_+} - \frac{1}{R_-}\right]$$
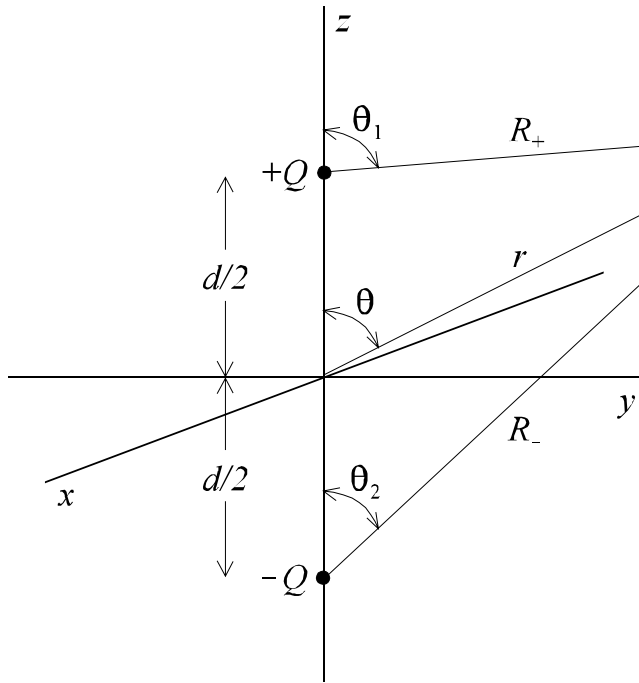
If the field point P is moved a large distance from the electric dipole (in what is called the *far field*, $r \gg d$) the lines connecting the two charges and the coordinate origin with the field point become nearly parallel.

$$\theta_1 \approx \theta_2 \approx \theta$$

$$R_+ \approx r - \frac{d}{2}\cos\theta$$

$$R_- \approx r + \frac{d}{2}\cos\theta$$

$$V \approx \frac{Q}{4\pi\epsilon_o}\left[\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right]$$

$$= \frac{Q}{4\pi\epsilon_o}\left[\frac{\left(r + \frac{d}{2}\cos\theta\right) - \left(r - \frac{d}{2}\cos\theta\right)}{r^2 - \frac{d^2}{2}\cos^2\theta}\right]$$

$$= \frac{Qd\cos\theta}{4\pi\epsilon_o r^2} \qquad \text{(Dipole far field potential, } r \gg d)$$

The electric field produced by the electric dipole is found by taking the gradient of the potential.

$$\boldsymbol{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\boldsymbol{a}_\theta\right]$$

$$= -\frac{Qd}{4\pi\epsilon_o}\left[\cos\theta\,\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\boldsymbol{a}_r + \frac{1}{r^3}\frac{\partial}{\partial\theta}(\cos\theta)\,\boldsymbol{a}_\theta\right]$$

$$= -\frac{Qd}{4\pi\epsilon_o}\left[\cos\theta\left(-\frac{2}{r^3}\right)\boldsymbol{a}_r + \frac{1}{r^3}(-\sin\theta)\,\boldsymbol{a}_\theta\right]$$

$$= \frac{Qd}{4\pi\epsilon_o r^3}\left[2\cos\theta\,\boldsymbol{a}_r + \sin\theta\,\boldsymbol{a}_\theta\right] \qquad \text{(Dipole electric field, far field, } r \gg d)$$

If the *vector dipole moment* is defined as

$$\boldsymbol{p} = p\boldsymbol{a}_p = Qd\,\boldsymbol{a}_p \qquad (\boldsymbol{a}_p \text{ points from } -Q \text{ to} +Q)$$

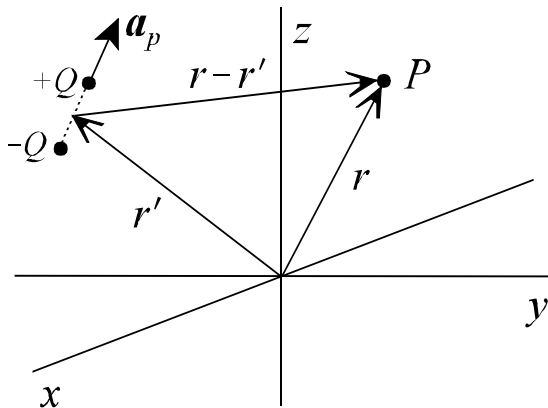the dipole potential and electric field may be written as

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_o r^2} = \frac{\boldsymbol{p}\cdot\boldsymbol{a}_r}{4\pi\epsilon_o r^2}$$

$$\boldsymbol{E} = \frac{p}{4\pi\epsilon_o r^3}\left[2\cos\theta\,\boldsymbol{a}_r + \sin\theta\,\boldsymbol{a}_\theta\right]$$

Note that the potential and electric field of the electric dipole decay faster than those of a point charge.

	V	E
point charge	$\sim r^{-1}$	$\sim r^{-2}$
electric dipole	$\sim r^{-2}$	$\sim r^{-3}$

For an arbitrarily located, arbitrarily oriented dipole, the potential can be written as

$$V = \frac{\boldsymbol{p}\cdot\left(\dfrac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)}{4\pi\epsilon_o |\boldsymbol{r}-\boldsymbol{r}'|^2}$$

$$= \frac{\boldsymbol{p}\cdot\left(\boldsymbol{r}-\boldsymbol{r}'\right)}{4\pi\epsilon_o |\boldsymbol{r}-\boldsymbol{r}'|^3}$$

$$(|\boldsymbol{r}-\boldsymbol{r}'| \gg d)$$

Energy Density in the Electric Field

The amount of work necessary to assemble a group of point charges equals the total energy (W_E) stored in the resulting electric field.

<u>Example</u> (3 point charges)

Given a system of 3 point charges, we can determine the total energy stored in the electric field of these point charges by determining the work performed to assemble the charge distribution. We first define V_{mn} as the absolute potential at P_m due to point charge Q_n.
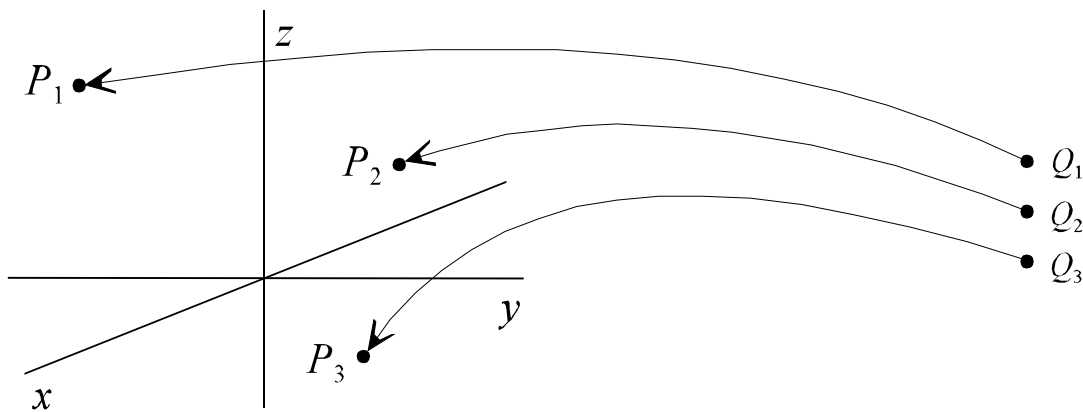
1. Bring Q_1 to P_1 (no energy required).
2. Bring Q_2 to P_2 (work = $Q_2 V_{21}$).
3. Bring Q_3 to P_3 (work = $Q_3 V_{31} + Q_3 V_{32}$).

$$W_E = 0 + (Q_2 V_{21}) + (Q_3 V_{31} + Q_3 V_{32}) \qquad (1)$$

If we reverse the order in which the charges are assembled, the total energy required is the same as before.

1. Bring Q_3 to P_3 (no energy required).
2. Bring Q_2 to P_2 (work = $Q_2 V_{23}$).
3. Bring Q_1 to P_1 (work = $Q_1 V_{12} + Q_1 V_{13}$).

$$W_E = 0 + (Q_2 V_{23}) + (Q_1 V_{12} + Q_1 V_{13}) \qquad (2)$$

Adding equations (1) and (2) gives

$$2W_E = Q_1(V_{12}+V_{13}) + Q_2(V_{21}+V_{23}) + Q_3(V_{31}+V_{32}) = Q_1V_1 + Q_2V_2 + Q_3V_3$$

where V_m = total absolute potential at P_m affecting Q_m.

$$W_E = \tfrac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3)$$

In general, for a system of N point charges, the total energy in the electric field is given by

$$W_E = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k$$

For line, surface or volume charge distributions, the discrete sum total energy formula above becomes a continuous sum (integral) over the respective charge distribution. The point charge term is replaced by the appropriate differential element of charge for a line, surface or volume distribution: $\rho_L dl$, $\rho_S ds$ or $\rho_V dv$. The overall potential acting on the point charge Q_k due to the other point charges (V_k) is replaced by the overall potential (V) acting on the differential element of charge due to the rest of the charge distribution. The total energy expressions become
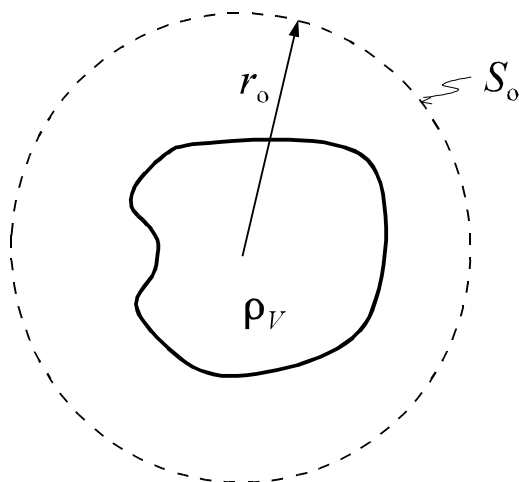
$$W_E = \frac{1}{2}\int \rho_L V\,dl \qquad \text{(line charge)}$$

$$W_E = \frac{1}{2}\iint \rho_S V\,ds \qquad \text{(surface charge)}$$

$$W_E = \frac{1}{2}\iiint \rho_V V\,dv \qquad \text{(volume charge)}$$

Total Energy in Terms of the Electric Field

If a volume charge distribution ρ_V of finite dimension is enclosed by a spherical surface S_o of radius r_o, the total energy associated with the charge is given by

$$W_E = \lim_{r_o \to \infty} \left[\frac{1}{2} \iiint_{V_o} \rho_V V \, dv \right] = \lim_{r_o \to \infty} \left[\frac{1}{2} \iiint_{V_o} (\nabla \cdot \boldsymbol{D}) V \, dv \right]$$

Using the following vector identity,

$$(\nabla \cdot \boldsymbol{F})f = \nabla \cdot (f\boldsymbol{F}) - \boldsymbol{F} \cdot \nabla f$$

the expression for the total energy may be written as

$$W_E = \lim_{r_o \to \infty} \left[\frac{1}{2} \iiint_{V_o} [\nabla \cdot (V\boldsymbol{D})] \, dv - \frac{1}{2} \iiint_{V_o} (\boldsymbol{D} \cdot \nabla V) \, dv \right]$$

If we apply the divergence theorem to the first integral, we find

$$W_E = \lim_{r_o \to \infty} \left[\frac{1}{2} \iint_{S_o} (V\boldsymbol{D}) \cdot d\boldsymbol{s} - \frac{1}{2} \iiint_{V_o} (\boldsymbol{D} \cdot \nabla V) \, dv \right]$$

For each equivalent point charge ($\rho_V \, dv$) that makes up the volume charge distribution, the potential contribution on S_o varies as r^{-1} and electric flux density (and electric field) contribution varies as r^{-2}. Thus, the product of the potential and electric flux density on the surface S_o varies as r^{-3}. Since the integration over the surface provides a multiplication factor of only r^2, the surface integral in the energy equation goes to zero on the surface S_o of infinite radius. This yields

$$W_E = -\frac{1}{2} \iiint (\boldsymbol{D} \cdot \nabla V) \, dv$$

where the integration is applied over all space. The divergence term in the integrand can be written in terms of the electric field as

$$\boldsymbol{E} = -\nabla V$$

such that the total energy (J) in the electric field is

$$W_E = \frac{1}{2} \iiint (\boldsymbol{D} \cdot \boldsymbol{E}) \, dv = \frac{1}{2} \iiint \epsilon_o (\boldsymbol{E} \cdot \boldsymbol{E}) \, dv$$

$$= \frac{1}{2} \iiint \epsilon_o E^2 \, dv$$

The total energy in the previous integral can be written as the integral of the electric field energy density (w_E) throughout the volume.

$$W_E = \iiint w_E \, dv$$

Thus, the energy density in an electric field is given by

$$w_E = \frac{1}{2} \epsilon_o E^2 \qquad (\text{J/m}^3)$$

Example (Energy density / total energy in an electric field)

Given $V = (x - y + xy + 2z)$ volts, determine the electrostatic energy stored in a cube of side 2m centered at the origin.

The electric field is found by taking the gradient of the potential function.

$$\boldsymbol{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial y}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right]$$

$$= -[(1+y)\boldsymbol{a}_x + (-1+x)\boldsymbol{a}_y + 2\boldsymbol{a}_z] \qquad (\text{V/m})$$

The energy density in the electric field is given by

$$w_E = \frac{1}{2}\epsilon_o E^2 \qquad (\text{W/m}^3)$$

$$E^2 = \boldsymbol{E}\cdot\boldsymbol{E} = E_x^2 + E_y^2 + E_z^2$$

$$= (y+1)^2 + (x-1)^2 + 2^2$$

$$= (y^2 + 2y + 1) + (x^2 - 2x + 1) + 4$$

$$= x^2 - 2x + y^2 + 2y + 6$$

$$w_E = \frac{\epsilon_o}{2}(x^2 - 2x + y^2 + 2y + 6)$$

The total energy within the defined cube is found by integrating the energy density throughout the cube.

$$W_E = \iiint w_E\, dv = \frac{\epsilon_o}{2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(x^2 - \overset{0}{\cancel{2x}} + y^2 + \overset{0}{\cancel{2y}} + 6)\,dx\,dy\,dz$$

(Odd integrands / symmetric limits)

$$W_E = \frac{\epsilon_o}{2} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (x^2 + y^2 + 6)\, dx\, dy\, dz$$

$$= \frac{\epsilon_o}{2} \left[\left(\frac{x^3}{3}\right)_{-1}^{1} (y)_{-1}^{1} (z)_{-1}^{1} + (x)_{-1}^{1} \left(\frac{y^3}{3}\right)_{-1}^{1} (z)_{-1}^{1} + 6\,(x)_{-1}^{1} (y)_{-1}^{1} (z)_{-1}^{1} \right]$$

$$= \frac{\epsilon_o}{2} \left[\frac{2}{3}(2)(2) + (2)\frac{2}{3}(2) + 6(2)(2)(2) \right] = \frac{80}{3}\epsilon_o$$

$$= 236 \text{ pJ}$$