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Biot–Savart Law

Problem 1.1 An 8 cm × 12 cm rectangular loop of wire is situated in the x–y plane with the center of the loop at the origin and its long sides parallel to the x-axis. The loop has a current of 50 A flowing with clockwise direction (when viewed from above). Determine the magnetic field at the center of the loop.

Solution: The total magnetic field is the vector sum of the individual fields of each of the four wire segments: $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4$. An expression for the magnetic field from a wire segment is given by:

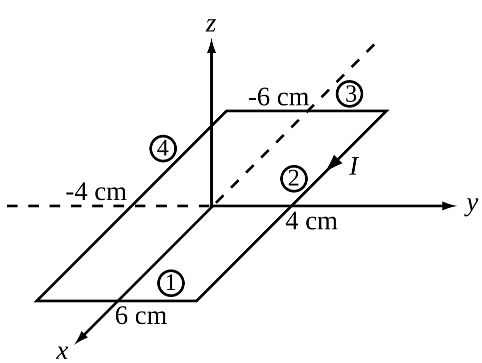

Figure P1.1: Problem 1.1.

For all segments shown in Fig. P1.1, the combination of the direction of the current and the right-hand rule gives the direction of the magnetic field as $-z$ direction at the origin. With $r = 6$ cm and $l = 8$ cm,

$$\begin{aligned}\mathbf{B}_1 &= -\hat{\mathbf{z}}\frac{\mu I l}{2\pi r\sqrt{4r^2 + l^2}} \\ &= -\hat{\mathbf{z}}\frac{4\pi \times 10^{-7} \times 50 \times 0.08}{2\pi \times 0.06 \times \sqrt{4 \times 0.06^2 + 0.08^2}} = -\hat{\mathbf{z}}9.24 \times 10^{-5} \quad \text{(T)}.\end{aligned}$$

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For segment 2, $r = 4$ cm and $l = 12$ cm,

$$\mathbf{B}_2 = -\hat{\mathbf{z}}\frac{\mu I l}{2\pi r\sqrt{4r^2 + l^2}}$$
$$= -\hat{\mathbf{z}}\frac{4\pi \times 10^{-7} \times 50 \times 0.12}{2\pi \times 0.04 \times \sqrt{4 \times 0.04^2 + 0.12^2}} = -\hat{\mathbf{z}}20.80 \times 10^{-5} \quad \text{(T)}.$$

Similarly,

$$\mathbf{B}_3 = -\hat{\mathbf{z}}9.24 \times 10^{-5} \quad \text{(T)}, \qquad \mathbf{B}_4 = -\hat{\mathbf{z}}20.80 \times 10^{-5} \quad \text{(T)}.$$

The total field is then $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4 = -\hat{\mathbf{z}}0.60$ (mT).

Problem 1.2

Develop an expression for the magnetic field $\mathbf{H}$ at an arbitrary point P due to the linear conductor defined by the geometry shown in Fig.2. . If the conductor extends between $z_1 = 3$ m and $z_2 = 7$ m and carries a current $I = 15$ A, find $\mathbf{H}$ at $P(2, \phi, 0)$.

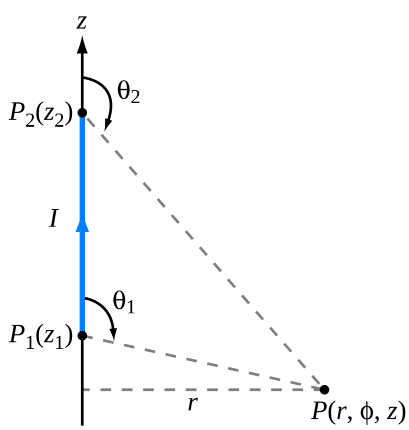

Fig 2.

Solution

the expressions for the cosines of the angles should be generalized to read as

$$\cos\theta_1 = \frac{z - z_1}{\sqrt{r^2 + (z - z_1)^2}}, \qquad \cos\theta_2 = \frac{z - z_2}{\sqrt{r^2 + (z - z_2)^2}}$$

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Plugging these expressions back into the magnetic field to give

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi r} \left(\frac{z - z_1}{\sqrt{r^2 + (z - z_1)^2}} - \frac{z - z_2}{\sqrt{r^2 + (z - z_2)^2}} \right).$$

For the specific geometry of Fig. P5.8,

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{15}{4\pi \times 2} \left[\frac{0-3}{\sqrt{3^2+2^2}} - \frac{0-7}{\sqrt{7^2+2^2}} \right] = \hat{\boldsymbol{\phi}} 77.4 \times 10^{-3} \text{ (A/m)} = \hat{\boldsymbol{\phi}} 77.4 \quad \text{(mA/m)}.$$

Problem 1.3 The loop shown in Fig. 3 consists of radial lines and segments of circles whose centers are at point P. Determine the magnetic field $\mathbf{H}$ at P in terms of a, b, θ, and I.

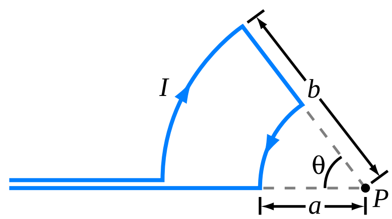

Fig.3

Solution: we denote the z-axis as passing out of the page through point P, the magnetic field pointing out of the page at P due to the current flowing in the outer arc is $\mathbf{H}_{\text{outer}} = -\hat{\mathbf{z}} I\theta/4\pi b$ and the field pointing out of the page at P due to the current flowing in the inner arc is $\mathbf{H}_{\text{inner}} = \hat{\mathbf{z}} I\theta/4\pi a$. The other wire segments do not contribute to the magnetic field at P. Therefore, the total field flowing directly out of the page at P is

$$\mathbf{H} = \mathbf{H}_{\text{outer}} + \mathbf{H}_{\text{inner}} = \hat{\mathbf{z}} \frac{I\theta}{4\pi} \left(\frac{1}{a} - \frac{1}{b} \right) = \hat{\mathbf{z}} \frac{I\theta(b-a)}{4\pi ab}.$$

Problem 1.4 An infinitely long, thin conducting sheet defined over the space $0 \le x \le w$ and $-\infty \le y \le \infty$ is carrying a current with a uniform surface current

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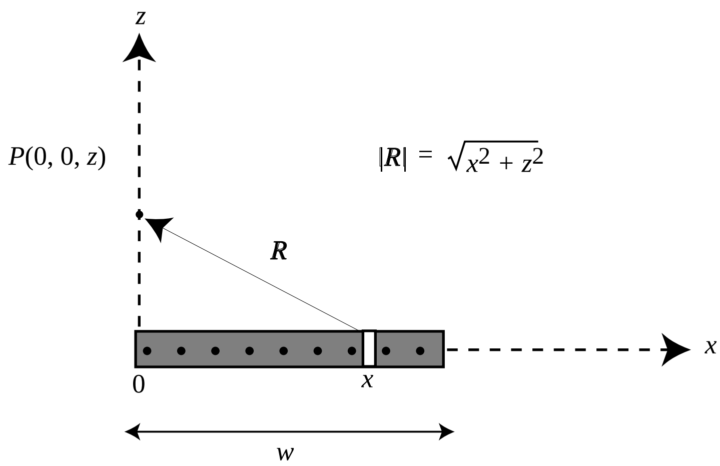

Fig 4: Conducting sheet of width w in x–y plane.

density $\mathbf{J}_s = \hat{\mathbf{y}}5$ (A/m). Obtain an expression for the magnetic field at point $P(0,0,z)$ in Cartesian coordinates.

Solution: The sheet can be considered to be a large number of infinitely long but narrow wires each dx wide lying next to each other, with each carrying a current $I_x = J_s\, dx$. The wire at a distance x from the origin is at a distance vector $\mathbf{R}$ from point P, with

$$\mathbf{R} = -\hat{\mathbf{x}}x + \hat{\mathbf{z}}z.$$

Equation below provides an expression for the magnetic field due to an infinitely long wire carrying a current I as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\hat{\boldsymbol{\phi}} I}{2\pi r}.$$

We now need to adapt this expression to the present situation by replacing I with $I_x = J_s\, dx$, replacing r with $R = (x^2 + z^2)^{1/2}$, as shown in Fig. 4, and by assigning the proper direction for the magnetic field. From the Biot–Savart law, the direction of $\mathbf{H}$ is governed by $\mathbf{l} \times \mathbf{R}$, where $\mathbf{l}$ is the direction of current flow. In the present case, $\mathbf{l}$ is in the $\hat{\mathbf{y}}$ direction. Hence, the direction of the field is

$$\frac{\mathbf{l} \times \mathbf{R}}{|\mathbf{l} \times \mathbf{R}|} = \frac{\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}x + \hat{\mathbf{z}}z)}{|\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}x + \hat{\mathbf{z}}z)|} = \frac{\hat{\mathbf{x}}z + \hat{\mathbf{z}}x}{(x^2 + z^2)^{1/2}}.$$

Therefore, the field $d\mathbf{H}$ due to the current I_x is

$$d\mathbf{H} = \frac{\hat{\mathbf{x}}z + \hat{\mathbf{z}}x}{(x^2+z^2)^{1/2}} \frac{I_x}{2\pi R} = \frac{(\hat{\mathbf{x}}z + \hat{\mathbf{z}}x) J_s \, dx}{2\pi(x^2+z^2)},$$

and the total field is

$$\begin{aligned}
\mathbf{H}(0,0,z) &= \int_{x=0}^{w} (\hat{\mathbf{x}}z + \hat{\mathbf{z}}x) \frac{J_s \, dx}{2\pi(x^2+z^2)} \\
&= \frac{J_s}{2\pi} \int_{x=0}^{w} (\hat{\mathbf{x}}z + \hat{\mathbf{z}}x) \frac{dx}{x^2+z^2} \\
&= \frac{J_s}{2\pi} \left(\hat{\mathbf{x}}z \int_{x=0}^{w} \frac{dx}{x^2+z^2} + \hat{\mathbf{z}} \int_{x=0}^{w} \frac{x\,dx}{x^2+z^2} \right) \\
&= \frac{J_s}{2\pi} \left(\hat{\mathbf{x}}z \left(\frac{1}{z} \tan^{-1}\left(\frac{x}{z}\right) \right) \Bigg|_{x=0}^{w} + \hat{\mathbf{z}} \left(\tfrac{1}{2} \ln(x^2+z^2) \right) \Big|_{x=0}^{w} \right) \\
&= \frac{5}{2\pi} \left[\hat{\mathbf{x}} 2\pi \tan^{-1}\left(\frac{w}{z}\right) + \hat{\mathbf{z}} \tfrac{1}{2} (\ln(w^2+z^2) - \ln(0+z^2)) \right] \quad \text{for } z \neq 0, \\
&= \frac{5}{2\pi} \left[\hat{\mathbf{x}} 2\pi \tan^{-1}\left(\frac{w}{z}\right) + \hat{\mathbf{z}} \tfrac{1}{2} \ln\left(\frac{w^2+z^2}{z^2}\right) \right] \quad \text{(A/m)} \quad \text{for } z \neq 0.
\end{aligned}$$

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Problem 1.5 An infinitely long wire carrying a 25-A current in the positive x-direction is placed along the x-axis in the vicinity of a 20-turn circular loop located in the x–y plane as shown in Fig. 5. If the magnetic field at the center of the loop is zero, what is the direction and magnitude of the current flowing in the loop?

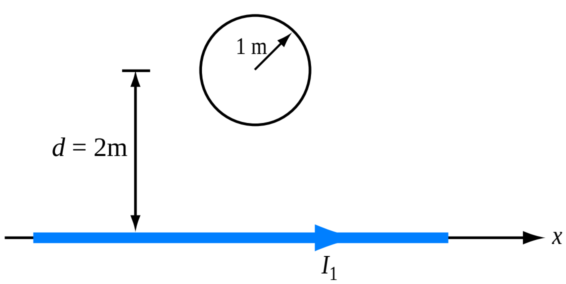

Figure 5: (a) Circular loop next to a linear current

Solution: The magnetic flux density at the center of the loop due to

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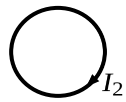

Figure 5: (b) Direction of I_2.

the wire is

$$\mathbf{B}_1 = \hat{\mathbf{z}}\,\frac{\mu_0}{2\pi d}\,I_1$$

where $\hat{\mathbf{z}}$ is out of the page. Since the net field is zero at the center of the loop, I_2 must be clockwise, as seen from above, in order to oppose I_1. The field due to I_2 is,

,

$$\mathbf{B} = \mu_0\mathbf{H} = -\hat{\mathbf{z}}\,\frac{\mu_0 N I_2}{2a}\,.$$

Equating the magnitudes of the two fields, we obtain the result

$$\frac{NI_2}{2a} = \frac{I_1}{2\pi d}\,,$$

or

$$I_2 = \frac{2aI_1}{2\pi N d} = \frac{1\times 25}{\pi\times 20\times 2} = 0.2\text{ A.}$$

Problem 1.6 Two infinitely long, parallel wires carry 6-A currents in opposite directions. Determine the magnetic flux density at point P in Fig. 6.

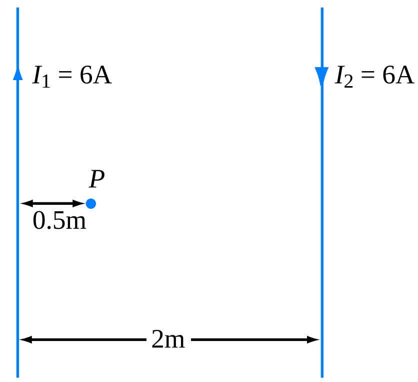

Fig 6: Arrangement for Problem

Solution:

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I_1}{2\pi(0.5)} + \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I_2}{2\pi(1.5)} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0}{\pi}\,(6+2) = \hat{\boldsymbol{\phi}}\,\frac{8\mu_0}{\pi} \quad \text{(T).}$$

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Problem 1.7 long, East-West oriented power cable carrying an unknown current I is at a height of 8 m above the Earth's surface. If the magnetic flux density recorded by a magnetic-field meter placed at the surface is 15 μT when the current is flowing through the cable and 20 μT when the current is zero, what is the magnitude of I?

Solution: The power cable is producing a magnetic flux density that opposes Earth's, own magnetic field. An East–West cable would produce a field whose direction at the surface is along North–South. The flux density due to the cable is

$$B = (20 - 15)\ \mu\text{T} = 5\mu\text{T}.$$

As a magnet, the Earth's field lines are directed from the South Pole to the North Pole inside the Earth and the opposite on the surface. Thus the lines at the surface are from North to South, which means that the field created by the cable is from South to North. Hence, by the right-hand rule, the current direction is toward the East. Its magnitude is obtained from

$$5\ \mu\text{T} = 5 \times 10^{-6} = \frac{\mu_0 I}{2\pi d} = \frac{4\pi \times 10^{-7} I}{2\pi \times 8},$$

which gives $I = 200$ A.

Problem 1.8 Two parallel, circular loops carrying a current of 40 A each are arranged as shown in Fig. 8. The first loop is situated in the x–y plane with its center at the origin and the second loop's center is at $z = 2$ m. If the two loops have the same radius $a = 3$ m, determine the magnetic field at:

(a) $z = 0$,

(b) $z = 1$ m,

(c) $z = 2$ m.

Solution: The magnetic field due to a circular loop is given for a loop in the x–y plane carrying a current I in the $+\hat{\boldsymbol{\phi}}$-direction. Considering that the bottom loop in Fig. 8 is in the x–y plane, but the current direction is along $-\hat{\boldsymbol{\phi}}$,

$$\mathbf{H}_1 = -\hat{\mathbf{z}} \frac{Ia^2}{2(a^2 + z^2)^{3/2}},$$

where z is the observation point along the z-axis. For the second loop, which is at a height of 2 m, we can use the same expression but z should be replaced with $(z - 2)$. Hence,

$$\mathbf{H}_2 = -\hat{\mathbf{z}} \frac{Ia^2}{2[a^2 + (z - 2)^2]^{3/2}}.$$

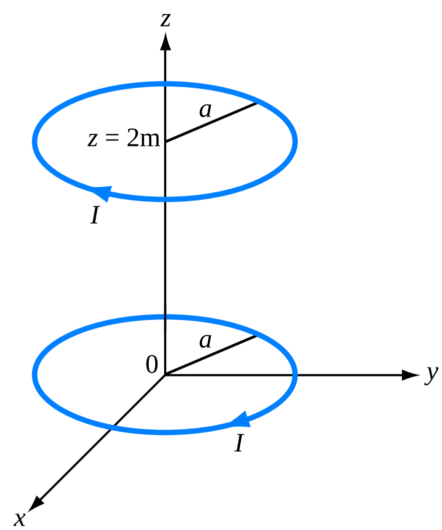

Fig 8: Parallel circular loops

The total field is

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = -\hat{\mathbf{z}}\frac{Ia^2}{2}\left[\frac{1}{(a^2+z^2)^{3/2}} + \frac{1}{[a^2+(z-2)^2]^{3/2}}\right] \text{ A/m.}$$

(a) At $z = 0$, and with $a = 3$ m and $I = 40$ A,

$$\mathbf{H} = -\hat{\mathbf{z}}\frac{40\times 9}{2}\left[\frac{1}{3^3} + \frac{1}{(9+4)^{3/2}}\right] = -\hat{\mathbf{z}}10.5 \text{ A/m.}$$

(b) At $z = 1$ m (midway between the loops):

$$\mathbf{H} = -\hat{\mathbf{z}}\frac{40\times 9}{2}\left[\frac{1}{(9+1)^{3/2}} + \frac{1}{(9+1)^{3/2}}\right] = -\hat{\mathbf{z}}11.38 \text{ A/m.}$$

(c) At $z = 2$ m, $\mathbf{H}$ should be the same as at $z = 0$. Thus,

$$\mathbf{H} = -\hat{\mathbf{z}}10.5 \text{ A/m.}$$