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Ampère's Law

Problem 2.1 Current I f ows along the positive z-direction in the inner conductor of a long coaxial cable and returns through the outer conductor. The inner conductor has radius a, and the inner and outer radii of the outer conductor are b and c, respectively.

(a) Determine the magnetic f eld in each of the following regions: $0 \le r \le a$, $a \le r \le b$, $b \le r \le c$, and $r \ge c$.

(b) Plot the magnitude of $\mathbf{H}$ as a function of r over the range from $r = 0$ to $r = 10$ cm, given that $I = 10$ A, $a = 2$ cm, $b = 4$ cm, and $c = 5$ cm.

Solution:

(a) The magnetic f eld in the region $r < a$,

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{rI}{2\pi a^2},$$

and in the region $a < r < b$,

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{2\pi r}.$$

The total area of the outer conductor is $A = \pi(c^2 - b^2)$ and the fraction of the area of the outer conductor enclosed by a circular contour centered at $r = 0$ in the region $b < r < c$ is

$$\frac{\pi(r^2 - b^2)}{\pi(c^2 - b^2)} = \frac{r^2 - b^2}{c^2 - b^2}.$$

The total current enclosed by a contour of radius r is therefore

$$I_{\text{enclosed}} = I\left(1 - \frac{r^2 - b^2}{c^2 - b^2}\right) = I\frac{c^2 - r^2}{c^2 - b^2},$$

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and the resulting magnetic f eld is

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I_{\text{enclosed}}}{2\pi r} = \hat{\boldsymbol{\phi}} \frac{I}{2\pi r} \left(\frac{c^2 - r^2}{c^2 - b^2} \right).$$

For $r > c$, the total enclosed current is zero: the total current f owing on the inner conductor is equal to the total current f owing on the outer conductor, but they are f owing in opposite directions. Therefore, $\mathbf{H} = 0$.

(b) See Fig.1

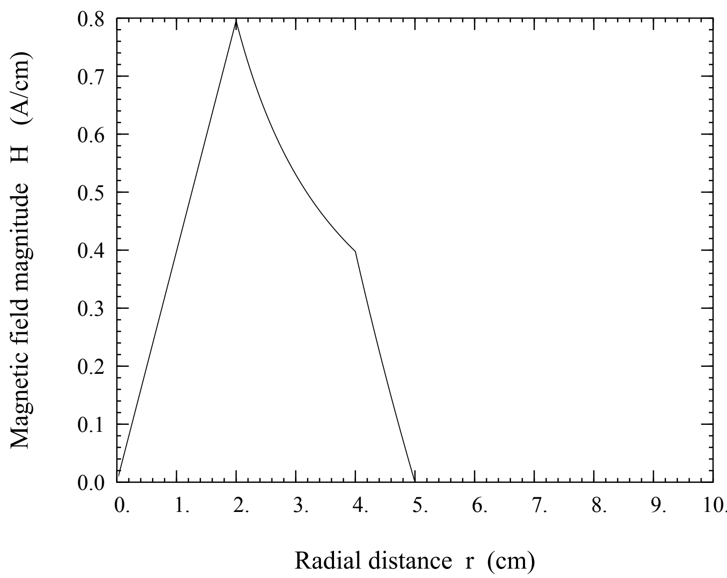

Fig 1.

Problem 2.2 long cylindrical conductor whose axis is coincident with the z-axis has a radius a and carries a current characterized by a current density $\mathbf{J} = \hat{\mathbf{z}} J_0 / r$, where J_0 is a constant and r is the radial distance from the cylinder's axis. Obtain an expression for the magnetic f eld $\mathbf{H}$ for (a) $0 \le r \le a$ and (b) $r > a$.

Solution:

(a) For $0 \le r_1 \le a$, the total current f owing within the contour C_1 is

$$I_1 = \iint \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{r=0}^{r_1} \left(\frac{\hat{\mathbf{z}} J_0}{r} \right) \cdot (\hat{\mathbf{z}} r \, dr \, d\phi) = 2\pi \int_{r=0}^{r_1} J_0 \, dr = 2\pi r_1 J_0.$$

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Therefore, since $I_1 = 2\pi r_1 H_1$, $H_1 = J_0$ within the wire and $\mathbf{H}_1 = \hat{\boldsymbol{\phi}} J_0$.

(b) For $r \geq a$, the total current f owing within the contour is the total current f owing within the wire:

$$I = \iint \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{r=0}^{a} \left(\frac{\hat{\mathbf{z}} J_0}{r} \right) \cdot (\hat{\mathbf{z}} r \, dr \, d\phi) = 2\pi \int_{r=0}^{a} J_0 \, dr = 2\pi a J_0.$$

Therefore, since $I = 2\pi r H_2$, $H_2 = J_0 a/r$ within the wire and $\mathbf{H}_2 = \hat{\boldsymbol{\phi}} J_0 (a/r)$.

Problem 2.3 Repeat Problem 2.2 for a current density $\mathbf{J} = \hat{\mathbf{z}} J_0 e^{-r}$.

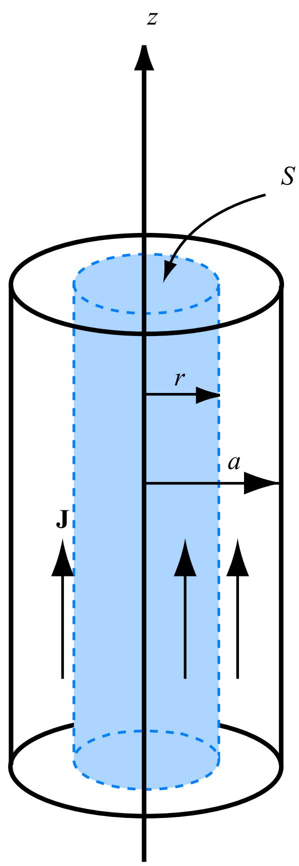

Fig 2: Cylindrical current.

Solution:

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(a) For $r \leq a$, Ampère's law is

$$\oint_c \mathbf{H} \cdot d\mathbf{l} = I = \int_S \mathbf{J} \cdot d\mathbf{s},$$
$$\hat{\boldsymbol{\phi}} H \cdot \hat{\boldsymbol{\phi}} 2\pi r = \int_0^r \mathbf{J} \cdot d\mathbf{s} = \int_0^r \hat{\mathbf{z}} J_0 e^{-r} \cdot \hat{\mathbf{z}} 2\pi r \, dr,$$
$$2\pi r H = 2\pi J_0 \int_0^r r e^{-r} \, dr$$
$$= 2\pi J_0 [-e^{-r}(r+1)]_0^r = 2\pi J_0 [1 - e^{-r}(r+1)].$$

Hence,

$$\mathbf{H} = \hat{\boldsymbol{\phi}} H = \hat{\boldsymbol{\phi}} \frac{J_0}{r} [1 - e^{-r}(r+1)], \qquad \text{for } r \leq a.$$

(b) For $r \geq a$,

$$2\pi r H = 2\pi J_0 [-e^{-r}(r+1)]_0^a = 2\pi J_0 [1 - e^{-a}(a+1)],$$
$$\mathbf{H} = \hat{\boldsymbol{\phi}} H = \hat{\boldsymbol{\phi}} \frac{J_0}{r} \left[1 - e^{-a}(a+1)\right], \qquad r \geq a.$$

Problem 2.4 In a certain conducting region, the magnetic f eld is given in cylindrical coordinates by

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{4}{r} [1 - (1+3r)e^{-3r}].$$

Find the current density **J**.

Solution:

$$\mathbf{J} = \nabla \times \mathbf{H} = \hat{\mathbf{z}} \frac{1}{r} \frac{\partial}{\partial r} \left(r \cdot \frac{4}{r} [1 - (1+3r)e^{-3r}] \right)$$
$$= \hat{\mathbf{z}} \frac{1}{r} [12e^{-2r}(1+2r) - 12e^{-2r}] = \hat{\mathbf{z}} 24 e^{-3r} \text{ A/m}^2$$
