Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Curl and Stokes's theorem

Problem 3.1 For the vector f eld $\mathbf{E} = \hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)$, calculate

(a) $\oint_C \mathbf{E} \cdot d\mathbf{l}$ around the triangular contour shown in Fig. 1.(a), and

(b) $\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s}$ over the area of the triangle.

Solution: In addition to the independent condition that $z = 0$, the three lines of the triangle are represented by the equations $y = 0$, $x = 1$, and $y = x$, respectively.

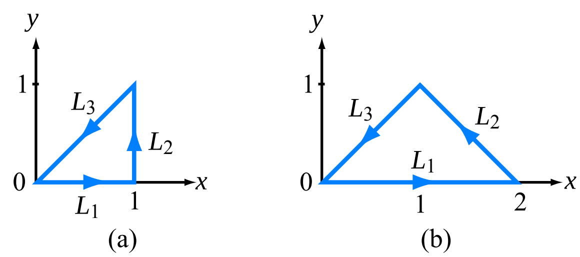

Fig 1

(a)

$$\oint \mathbf{E} \cdot d\mathbf{l} = L_1 + L_2 + L_3,$$

$$L_1 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$

$$= \int_{x=0}^{1} (xy)|_{y=0,z=0}\,dx - \int_{y=0}^{0} (x^2 + 2y^2)|_{z=0}\,dy + \int_{z=0}^{0} (0)|_{y=0}\,dz = 0,$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

$$L_2 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=1}^{1} (xy)|_{z=0}\,dx - \int_{y=0}^{1} \left(x^2+2y^2\right)\Big|_{x=1,z=0} dy + \int_{z=0}^{0} (0)|_{x=1}\,dz$$
$$= 0 - \left(y + \frac{2y^3}{3}\right)\Bigg|_{y=0}^{1} + 0 = \frac{-5}{3},$$
$$L_3 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=1}^{0} (xy)|_{y=x,\ z=0}\,dx - \int_{y=1}^{0} \left(x^2+2y^2\right)\Big|_{x=y,\ z=0} dy + \int_{z=0}^{0} (0)|_{y=x}\,dz$$
$$= \left(\frac{x^3}{3}\right)\Bigg|_{x=1}^{0} - \left(y^3\right)\Big|_{y=1}^{0} + 0 = \frac{2}{3}.$$

Therefore,
$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 - \frac{5}{3} + \frac{2}{3} = -1.$$

(b) $\nabla \times \mathbf{E} = -\hat{\mathbf{z}}3x$, so that
$$\iint \nabla \times \mathbf{E} \cdot d\mathbf{s} = \int_{x=0}^{1}\int_{y=0}^{x} \left((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}\,dy\,dx)\right)|_{z=0}$$
$$= -\int_{x=0}^{1}\int_{y=0}^{x} 3x\,dy\,dx = -\int_{x=0}^{1} 3x(x-0)\,dx = -\left(x^3\right)\Big|_0^1 = -1.$$

Problem 3.2 Repeat Problem 3.1 for the contour shown in Fig. 1. b.

Solution: In addition to the independent condition that $z = 0$, the three lines of the triangle are represented by the equations $y = 0$, $y = 2 - x$, and $y = x$, respectively.

(a)
$$\oint \mathbf{E} \cdot d\mathbf{l} = L_1 + L_2 + L_3,$$
$$L_1 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=0}^{2} (xy)|_{y=0,z=0}\,dx - \int_{y=0}^{0} \left(x^2+2y^2\right)\Big|_{z=0} dy + \int_{z=0}^{0} (0)|_{y=0}\,dz = 0,$$
$$L_2 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=2}^{1} (xy)|_{z=0,y=2-x}\,dx - \int_{y=0}^{1} \left(x^2+2y^2\right)\Big|_{x=2-y,z=0} dy + \int_{z=0}^{0} (0)|_{y=2-x}\,dz$$
$$= \left(x^2 - \frac{x^3}{3}\right)\Bigg|_{x=2}^{1} - \left(4y - 2y^2 + y^3\right)\Big|_{y=0}^{1} + 0 = \frac{-11}{3},$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

$$L_3 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$
$$= \int_{x=1}^{0} (xy)\big|_{y=x,z=0}\,dx - \int_{y=1}^{0} \left(x^2+2y^2\right)\big|_{x=y,z=0}\,dy + \int_{z=0}^{0} (0)\big|_{y=x}\,dz$$
$$= \left(\frac{x^3}{3}\right)\bigg|_{x=1}^{0} - \left(y^3\right)\big|_{y=1}^{0} + 0 = \frac{2}{3}.$$

Therefore,

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0 - \frac{11}{3} + \frac{2}{3} = -3.$$

(b) $\nabla\times\mathbf{E} = -\hat{\mathbf{z}}3x$, so that

$$\iint \nabla\times\mathbf{E}\cdot d\mathbf{s} = \int_{x=0}^{1}\int_{y=0}^{x} ((-\hat{\mathbf{z}}3x)\cdot(\hat{\mathbf{z}}\,dy\,dx))\big|_{z=0}$$
$$+ \int_{x=1}^{2}\int_{y=0}^{2-x} ((-\hat{\mathbf{z}}3x)\cdot(\hat{\mathbf{z}}\,dy\,dx))\big|_{z=0}$$
$$= -\int_{x=0}^{1}\int_{y=0}^{x} 3x\,dy\,dx - \int_{x=1}^{2}\int_{y=0}^{2-x} 3x\,dy\,dx$$
$$= -\int_{x=0}^{1} 3x(x-0)\,dx - \int_{x=1}^{2} 3x((2-x)-0)\,dx$$
$$= -\left(x^3\right)\big|_0^1 - \left(3x^2 - x^3\right)\big|_{x=1}^{2} = -3.$$

Problem 3.3 Verify Stokes's theorem for the vector f eld $\mathbf{B} = (\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$ by evaluating:

(a) $\oint_C \mathbf{B}\cdot d\mathbf{l}$ over the semicircular contour shown in Fig. 2. a, and

(b) $\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s}$ over the surface of the semicircle.

Solution:

(a)

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_{L_1} \mathbf{B}\cdot d\mathbf{l} + \int_{L_2} \mathbf{B}\cdot d\mathbf{l} + \int_{L_3} \mathbf{B}\cdot d\mathbf{l},$$
$$\mathbf{B}\cdot d\mathbf{l} = (\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$
$$\int_{L_1} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=0}^{2} r\cos\phi\,dr\right)\bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$
$$= \left(\tfrac{1}{2}r^2\right)\big|_{r=0}^{2} + 0 = 2,$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

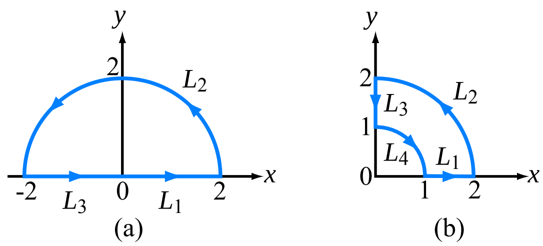

Fig 2.

$$\int_{L_2} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\, dr\right)\Big|_{z=0} + \left(\int_{\phi=0}^{\pi} r\sin\phi\, d\phi\right)\Big|_{r=2,\ z=0}$$
$$= 0 + (-2\cos\phi)|_{\phi=0}^{\pi} = 4,$$
$$\int_{L_3} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{0} r\cos\phi\, dr\right)\Big|_{\phi=\pi, z=0} + \left(\int_{\phi=\pi}^{\pi} r\sin\phi\, d\phi\right)\Big|_{z=0}$$
$$= \left(-\tfrac{1}{2}r^2\right)\Big|_{r=2}^{0} + 0 = 2,$$
$$\oint \mathbf{B}\cdot d\mathbf{l} = 2+4+2 = 8.$$

(b)

$$\nabla\times\mathbf{B} = \nabla\times(\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$$
$$= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}(\sin\phi)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi) - \frac{\partial}{\partial r}0\right)$$
$$+\hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi)) - \frac{\partial}{\partial\phi}(r\cos\phi)\right)$$
$$= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}(\sin\phi + (r\sin\phi)) = \hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right),$$
$$\iint \nabla\times\mathbf{B}\cdot d\mathbf{s} = \int_{\phi=0}^{\pi}\int_{r=0}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}} r\, dr\, d\phi)$$
$$= \int_{\phi=0}^{\pi}\int_{r=0}^{2}\sin\phi(r+1)\, dr\, d\phi = \left(\left(-\cos\phi(\tfrac{1}{2}r^2+r)\right)\Big|_{r=0}^{2}\right)\Big|_{\phi=0}^{\pi} = 8.$$

Problem 3.4 Repeat Problem 3.3 for the contour shown in Fig. 2. b.

Solution:

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

(a)

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_{L_1}\mathbf{B}\cdot d\mathbf{l} + \int_{L_2}\mathbf{B}\cdot d\mathbf{l} + \int_{L_3}\mathbf{B}\cdot d\mathbf{l} + \int_{L_4}\mathbf{B}\cdot d\mathbf{l},$$

$$\mathbf{B}\cdot d\mathbf{l} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}} r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$

$$\int_{L_1}\mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=1}^{2} r\cos\phi\,dr\right)\bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(\tfrac{1}{2}r^2\right)\Big|_{r=1}^{2} + 0 = \frac{3}{2},$$

$$\int_{L_2}\mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=0}^{\pi/} r\sin\phi\,d\phi\right)\bigg|_{r=2,\ z=0}$$

$$= 0 + (-2\cos\phi)\Big|_{\phi=0}^{\pi/2} = 2,$$

$$\int_{L_3}\mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{1} r\cos\phi\,dr\right)\bigg|_{\phi=\pi/2,\ z=0} + \left(\int_{\phi=\pi/2}^{\pi/2} r\sin\phi\,d\phi\right)\bigg|_{z=0} = 0,$$

$$\int_{L_4}\mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=1}^{1} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=\pi/2}^{0} r\sin\phi\,d\phi\right)\bigg|_{r=1,\ z=0}$$

$$= 0 + (-\cos\phi)\Big|_{\phi=\pi/2}^{0} = -1,$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = \frac{3}{2} + 2 + 0 - 1 = \frac{5}{2}.$$

(b)

$$\nabla\times\mathbf{B} = \nabla\times(\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$$

$$= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}(\sin\phi)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi) - \frac{\partial}{\partial r}0\right)$$

$$+ \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi)) - \frac{\partial}{\partial\phi}(r\cos\phi)\right)$$

$$= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}(\sin\phi + (r\sin\phi)) = \hat{\mathbf{z}}\sin\phi\left(1 + \frac{1}{r}\right),$$

$$\iint \nabla\times\mathbf{B}\cdot d\mathbf{s} = \int_{\phi=0}^{\pi/2}\int_{r=1}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}} r\,dr\,d\phi)$$

$$= \int_{\phi=0}^{\pi/2}\int_{r=1}^{2}\sin\phi(r+1)\,dr\,d\phi$$

$$= \left(\left(-\cos\phi(\tfrac{1}{2}r^2 + r)\right)\Big|_{r=1}^{2}\right)\bigg|_{\phi=0}^{\pi/2} = \frac{5}{2}.$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Problem 3.5 Verify Stokes's Theorem for the vector field $\mathbf{A} = \hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta$ by evaluating it on the hemisphere of unit radius.

Solution:

$$\mathbf{A} = \hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi.$$

Hence, $A_R = \cos\theta,\ A_\theta = 0,\ A_\phi = \sin\theta$.

$$\begin{aligned}
\nabla \times \mathbf{A} &= \hat{\mathbf{R}}\,\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\right) - \hat{\boldsymbol{\theta}}\,\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \hat{\boldsymbol{\phi}}\,\frac{1}{R}\frac{\partial A_R}{\partial\theta} \\
&= \hat{\mathbf{R}}\,\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta) - \hat{\boldsymbol{\theta}}\,\frac{1}{R}\frac{\partial}{\partial R}(R\sin\theta) - \hat{\boldsymbol{\phi}}\,\frac{1}{R}\frac{\partial}{\partial\theta}(\cos\theta) \\
&= \hat{\mathbf{R}}\,\frac{2\cos\theta}{R} - \hat{\boldsymbol{\theta}}\,\frac{\sin\theta}{R} + \hat{\boldsymbol{\phi}}\,\frac{\sin\theta}{R}.
\end{aligned}$$

For the hemispherical surface, $d\mathbf{s} = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$.

$$\begin{aligned}
&\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}(\nabla\times\mathbf{A})\cdot d\mathbf{s} \\
&= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left(\frac{\hat{\mathbf{R}}2\cos\theta}{R} - \hat{\boldsymbol{\theta}}\,\frac{\sin\theta}{R} + \hat{\boldsymbol{\phi}}\,\frac{\sin\theta}{R}\right)\cdot\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi\bigg|_{R=1} \\
&= 4\pi R\left.\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2}\right|_{R=1} = 2\pi.
\end{aligned}$$

The contour C is the circle in the x–y plane bounding the hemispherical surface.

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_{\phi=0}^{2\pi}(\hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta)\cdot\hat{\boldsymbol{\phi}}R\,d\phi\bigg|_{\substack{\theta=\pi/2\\R=1}} = R\sin\theta\int_0^{2\pi}d\phi\bigg|_{\substack{\theta=\pi/2\\R=1}} = 2\pi.$$

Problem 3.6 Determine if each of the following vector fields is solenoidal, conservative, or both:

(a) $\mathbf{A} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y2xy$,
(b) $\mathbf{B} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z$,
(c) $\mathbf{C} = \hat{\mathbf{r}}(\sin\phi)/r^2 + \hat{\boldsymbol{\phi}}(\cos\phi)/r^2$,
(d) $\mathbf{D} = \hat{\mathbf{R}}/R$,
(e) $\mathbf{E} = \hat{\mathbf{r}}\left(3 - \frac{r}{1+r}\right) + \hat{\mathbf{z}}z$,
(f) $\mathbf{F} = (\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)/(x^2+y^2)$,
(g) $\mathbf{G} = \hat{\mathbf{x}}(x^2+z^2) - \hat{\mathbf{y}}(y^2+x^2) - \hat{\mathbf{z}}(y^2+z^2)$,
(h) $\mathbf{H} = \hat{\mathbf{R}}(Re^{-R})$.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

Solution:

(a)

$$\nabla\cdot\mathbf{A} = \nabla\cdot(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}2xy) = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}2xy = 2x - 2x = 0,$$

$$\begin{aligned}\nabla\times\mathbf{A} &= \nabla\times(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}2xy)\\ &= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}(-2xy)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}0\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2)\right)\\ &= \hat{\mathbf{x}}0 + \hat{\mathbf{y}}0 - \hat{\mathbf{z}}(2y) \neq 0.\end{aligned}$$

The f eld **A** is solenoidal but not conservative.

(b)

$$\nabla\cdot\mathbf{B} = \nabla\cdot(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z) = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}2z = 2x - 2y + 2 \neq 0,$$

$$\begin{aligned}\nabla\times\mathbf{B} &= \nabla\times(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z)\\ &= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z}(-y^2)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(2z)\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(x^2)\right)\\ &= \hat{\mathbf{x}}0 + \hat{\mathbf{y}}0 + \hat{\mathbf{z}}0.\end{aligned}$$

The f eld **B** is conservative but not solenoidal.

(c)

$$\begin{aligned}\nabla\cdot\mathbf{C} &= \nabla\cdot\left(\hat{\mathbf{r}}\frac{\sin\phi}{r^2} + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right)\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\sin\phi}{r^2}\right)\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{r^2}\right) + \frac{\partial}{\partial z}0\\ &= \frac{-\sin\phi}{r^3} + \frac{-\sin\phi}{r^3} + 0 = \frac{-2\sin\phi}{r^3},\end{aligned}$$

$$\begin{aligned}\nabla\times\mathbf{C} &= \nabla\times\left(\hat{\mathbf{r}}\frac{\sin\phi}{r^2} + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right)\\ &= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}\left(\frac{\cos\phi}{r^2}\right)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}\left(\frac{\sin\phi}{r^2}\right) - \frac{\partial}{\partial r}0\right)\\ &\quad + \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\left(\frac{\cos\phi}{r^2}\right)\right) - \frac{\partial}{\partial\phi}\left(\frac{\sin\phi}{r^2}\right)\right)\\ &= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}\left(-\left(\frac{\cos\phi}{r^2}\right) - \left(\frac{\cos\phi}{r^2}\right)\right) = \hat{\mathbf{z}}\frac{-2\cos\phi}{r^3}.\end{aligned}$$

The f eld **C** is neither solenoidal nor conservative.

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

(d)

$$\nabla\cdot\mathbf{D} = \nabla\cdot\left(\frac{\hat{\mathbf{R}}}{R}\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\left(\frac{1}{R}\right)\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(0\sin\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}0 = \frac{1}{R^2},$$

$$\nabla\times\mathbf{D} = \nabla\times\left(\frac{\hat{\mathbf{R}}}{R}\right)$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(0\sin\theta) - \frac{\partial}{\partial\phi}0\right) + \hat{\boldsymbol{\theta}}\frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\right) - \frac{\partial}{\partial R}(R(0))\right)$$

$$+ \hat{\boldsymbol{\phi}}\frac{1}{R}\left(\frac{\partial}{\partial R}(R(0)) - \frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\right) = \hat{\mathbf{r}}0 + \hat{\boldsymbol{\theta}}0 + \hat{\boldsymbol{\phi}}0.$$

The f eld **D** is conservative but not solenoidal.

(e)

$$\mathbf{E} = \hat{\mathbf{r}}\left(3 - \frac{r}{1+r}\right) + \hat{\mathbf{z}}z,$$

$$\nabla\cdot\mathbf{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_{\mathrm{r}}) + \frac{1}{r}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(3r - \frac{r^2}{1+r}\right) + 1$$

$$= \frac{1}{r}\left[3 - \frac{2r}{1+r} + \frac{r^2}{(1+r)^2}\right] + 1$$

$$= \frac{1}{r}\left[\frac{3+3r^2+6r-2r-2r^2+r^2}{(1+r)^2}\right] + 1 = \frac{2r^2+4r+3}{r(1+r)^2} + 1 \neq 0,$$

$$\nabla\times\mathbf{E} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial E_z}{\partial\phi} - \frac{\partial E_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right) + \hat{\mathbf{z}}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi) - \frac{1}{r}\frac{\partial E_r}{\partial\phi}\right) = 0.$$

Hence, **E** is conservative, but not solenoidal.

(f)

$$\mathbf{F} = \frac{\hat{\mathbf{x}}y + \hat{\mathbf{y}}x}{x^2+y^2} = \hat{\mathbf{x}}\frac{y}{x^2+y^2} + \hat{\mathbf{y}}\frac{x}{x^2+y^2},$$

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{-2xy}{(x^2+y^2)^2} + \frac{-2xy}{(x^2+y^2)^2} \neq 0,$$

Dr. Ahmed Abdolkhalig
The University of Tobruk
Department of Electrical Engineering
www.ahmed.ucoz.org

$$\begin{aligned}\nabla\times\mathbf{F} &= \hat{\mathbf{x}}(0-0)+\hat{\mathbf{y}}(0-0)+\hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]\\ &= \hat{\mathbf{z}}\left(\frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2}-\frac{1}{x^2+y^2}+\frac{2y^2}{(x^2+y^2)^2}\right)\\ &= \hat{\mathbf{z}}\,\frac{2(y^2-x^2)}{(x^2+y^2)^2}\neq 0.\end{aligned}$$

Hence, **F** is neither solenoidal nor conservative.

(g)

$$\begin{aligned}\mathbf{G} &= \hat{\mathbf{x}}(x^2+z^2)-\hat{\mathbf{y}}(y^2+x^2)-\hat{\mathbf{z}}(y^2+z^2),\\ \nabla\cdot\mathbf{G} &= \frac{\partial}{\partial x}(x^2+z^2)-\frac{\partial}{\partial y}(y^2+x^2)-\frac{\partial}{\partial z}(y^2+z^2)\\ &= 2x-2y-2z\neq 0,\\ \nabla\times\mathbf{G} &= \hat{\mathbf{x}}\left(-\frac{\partial}{\partial y}(y^2+z^2)+\frac{\partial}{\partial z}(y^2+x^2)\right)+\hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2+z^2)+\frac{\partial}{\partial x}(y^2+z^2)\right)\\ &\quad+\hat{\mathbf{z}}\left(-\frac{\partial}{\partial x}(y^2+x^2)-\frac{\partial}{\partial y}(x^2+z^2)\right)\\ &= -\hat{\mathbf{x}}\,2y+\hat{\mathbf{y}}\,2z-\hat{\mathbf{z}}\,2x\neq 0.\end{aligned}$$

Hence, **G** is neither solenoidal nor conservative.

(h)

$$\begin{aligned}\mathbf{H} &= \hat{\mathbf{R}}(Re^{-R}),\\ \nabla\cdot\mathbf{H} &= \frac{1}{R^2}\frac{\partial}{\partial R}(R^3e^{-R})=\frac{1}{R^2}(3R^2e^{-R}-R^3e^{-R})=e^{-R}(3-R)\neq 0,\\ \nabla\times\mathbf{H} &= 0.\end{aligned}$$

Hence, **H** is conservative, but not solenoidal.