Q1-a Vector $\mathbf{A}$ starts at point $(1, -1, -3)$ and ends at point $(2, -1, 0)$. Find a unit vector in the direction of $\mathbf{A}$.

Solution:

$$\mathbf{A} = \hat{\mathbf{x}}(2-1) + \hat{\mathbf{y}}(-1-(-1)) + \hat{\mathbf{z}}(0-(-3)) = \hat{\mathbf{x}} + \hat{\mathbf{z}}3,$$

$$|\mathbf{A}| = \sqrt{1+9} = 3.16,$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}} + \hat{\mathbf{z}}3}{3.16} = \hat{\mathbf{x}}0.32 + \hat{\mathbf{z}}0.95.$$

Q1-b Given vectors $\mathbf{A} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}$, $\mathbf{B} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3$, and $\mathbf{C} = \hat{\mathbf{x}}4 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2$, show that $\mathbf{C}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

Solution:

$$\mathbf{A} \cdot \mathbf{C} = (\hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}) \cdot (\hat{\mathbf{x}}4 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2) = 8 - 6 - 2 = 0,$$

$$\mathbf{B} \cdot \mathbf{C} = (\hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3) \cdot (\hat{\mathbf{x}}4 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2) = 8 - 2 - 6 = 0.$$

Q2 Use the appropriate expression for the differential surface area ds to determine the area of each of the following surfaces:

(a) $r = 3;\ 0 \le \phi \le \pi/3;\ -2 \le z \le 2,$

(b) $2 \le r \le 5;\ \pi/2 \le \phi \le \pi;\ z = 0,$

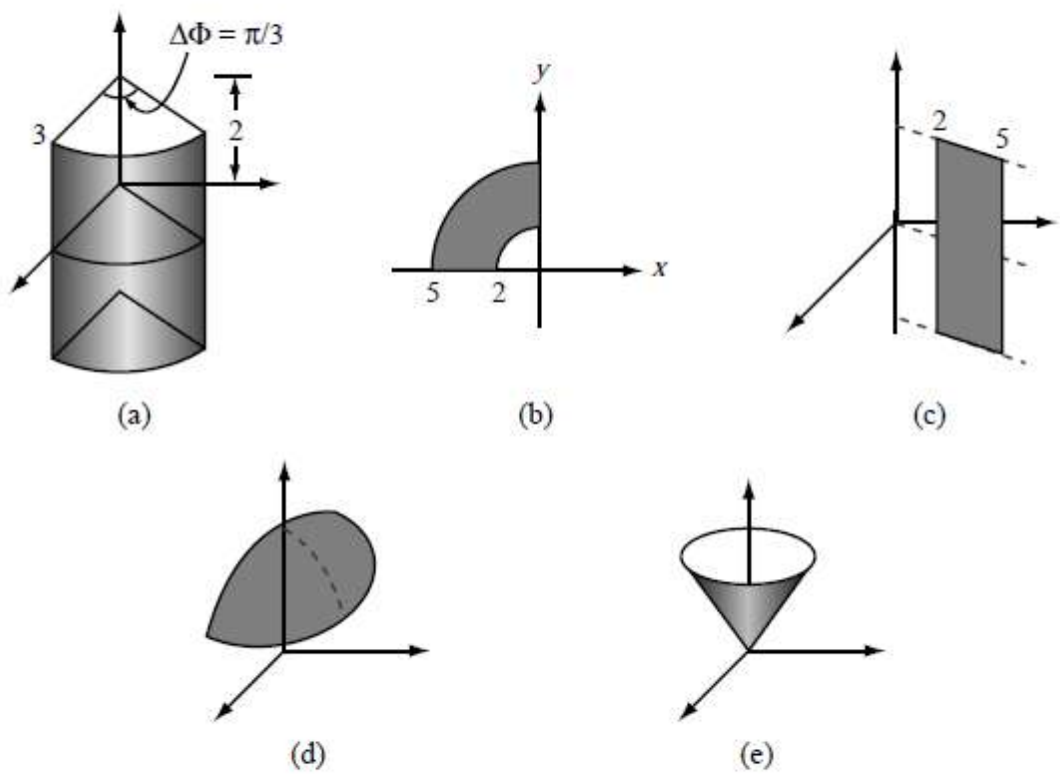
(c) $2 \le r \le 5;\ \phi = \pi/4;\ -2 \le z \le 2,$

(d) $R = 2;\ 0 \le \theta \le \pi/3;\ 0 \le \phi \le \pi,$

(e) $0 \le R \le 5;\ \theta = \pi/3;\ 0 \le \phi \le 2\pi.$

Also sketch the outlines of each of the surfaces.

Solution:



(a) (b) (c) (d) (e)

(a) Using Eq.

$$A = \int_{z=-2}^{2} \int_{\phi=0}^{\pi/3} (r)|_{r=3}\, d\phi\, dz = \left((3\phi z)\Big|_{\phi=0}^{\pi/3} \right)\Big|_{z=-2}^{2} = 4\pi.$$

(b) Using Eq.

$$A=\int_{r=2}^{5}\int_{\phi=\pi/2}^{\pi}(r)|_{z=0}\,d\phi\,dr=\left(\left(\tfrac{1}{2}r^2\phi\right)\Big|_{r=2}^{5}\right)\Big|_{\phi=\pi/2}^{\pi}=\frac{21\pi}{4}.$$

(c) Using Eq.

$$A=\int_{z=-2}^{2}\int_{r=2}^{5}(1)|_{\phi=\pi/4}\,dr\,dz=\left((rz)\big|_{z=-2}^{2}\right)\Big|_{r=2}^{5}=12.$$

(d) Using Eq.

$$A=\int_{\theta=0}^{\pi/3}\int_{\phi=0}^{\pi}(R^2\sin\theta)\big|_{R=2}\,d\phi\,d\theta=\left(\left(-4\phi\cos\theta\right)\big|_{\theta=0}^{\pi/3}\right)\Big|_{\phi=0}^{\pi}=2\pi.$$

(e) Using Eq.

$$A=\int_{R=0}^{5}\int_{\phi=0}^{2\pi}(R\sin\theta)|_{\theta=\pi/3}\,d\phi\,dR=\left(\left(\tfrac{1}{2}R^2\phi\sin\frac{\pi}{3}\right)\Big|_{\phi=0}^{2\pi}\right)\Big|_{R=0}^{5}=\frac{25\sqrt{3}\pi}{2}.$$

Q3 An infinitely long cylindrical shell extending between $r=1$ m and $r=3$ m contains a uniform charge density ρ_{v0}. Apply Gauss's law to find $\mathbf{D}$ in all regions.

Solution: For $r<1$ m, $\mathbf{D}=0$.
For $1\le r\le 3$ m,

$$\oint_S \hat{\mathbf{r}}\,D_r\cdot d\mathbf{s}=Q,$$

$$D_r\cdot 2\pi rL=\rho_{v0}\cdot\pi L(r^2-1^2),$$

$$\mathbf{D}=\hat{\mathbf{r}}\,D_r=\hat{\mathbf{r}}\,\frac{\rho_{v0}\pi L(r^2-1)}{2\pi rL}=\hat{\mathbf{r}}\,\frac{\rho_{v0}(r^2-1)}{2r},\qquad 1\le r\le 3\text{ m}.$$

For $r\ge 3$ m,

$$D_r\cdot 2\pi rL=\rho_{v0}\pi L(3^2-1^2)=8\rho_{v0}\pi L,$$

$$\mathbf{D}=\hat{\mathbf{r}}\,D_r=\hat{\mathbf{r}}\,\frac{4\rho_{v0}}{r},\qquad r\ge 3\text{ m}.$$