Tutorial#1

Chapter1

Section-1: Vector Algebra

Problem.1 Vector **A** starts at point $(1,-1,-3)$ and ends at point $(2,-1,0)$. Find a unit vector in the direction of **A**.

Solution:

$$\mathbf{A} = \hat{\mathbf{x}}(2-1)+\hat{\mathbf{y}}(-1-(-1))+\hat{\mathbf{z}}(0-(-3)) = \hat{\mathbf{x}}+\hat{\mathbf{z}}3,$$
$$|\mathbf{A}| = \sqrt{1+9} = 3.16,$$
$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}}+\hat{\mathbf{z}}3}{3.16} = \hat{\mathbf{x}}\,0.32+\hat{\mathbf{z}}\,0.95.$$

Problem.2 Given vectors $\mathbf{A} = \hat{\mathbf{x}}2-\hat{\mathbf{y}}3+\hat{\mathbf{z}}$, $\mathbf{B} = \hat{\mathbf{x}}2-\hat{\mathbf{y}}+\hat{\mathbf{z}}3$, and $\mathbf{C} = \hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2$, show that **C** is perpendicular to both **A** and **B**.

Solution:

$$\mathbf{A}\cdot\mathbf{C} = (\hat{\mathbf{x}}2-\hat{\mathbf{y}}3+\hat{\mathbf{z}})\cdot(\hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2) = 8-6-2 = 0,$$
$$\mathbf{B}\cdot\mathbf{C} = (\hat{\mathbf{x}}2-\hat{\mathbf{y}}+\hat{\mathbf{z}}3)\cdot(\hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2) = 8-2-6 = 0.$$

Problem.3 In Cartesian coordinates, the three corners of a triangle are $P_1(0,4,4)$, $P_2(4,-4,4)$, and $P_3(2,2,-4)$. Find the area of the triangle.

Solution: Let $\mathbf{B} = \overrightarrow{P_1P_2} = \hat{\mathbf{x}}4-\hat{\mathbf{y}}8$ and $\mathbf{C} = \overrightarrow{P_1P_3} = \hat{\mathbf{x}}2-\hat{\mathbf{y}}2-\hat{\mathbf{z}}8$ represent two sides of the triangle. Since the magnitude of the cross product is the area of the parallelogram (see the def nition of cross product), half of this is the area of the triangle:

$$A = \tfrac{1}{2}|\mathbf{B}\times\mathbf{C}| = \tfrac{1}{2}|(\hat{\mathbf{x}}4-\hat{\mathbf{y}}8)\times(\hat{\mathbf{x}}2-\hat{\mathbf{y}}2-\hat{\mathbf{z}}8)|$$
$$= \tfrac{1}{2}|\hat{\mathbf{x}}(-8)(-8)+\hat{\mathbf{y}}(-(4)(-8))+\hat{\mathbf{z}}(4(-2)-(-8)2)|$$
$$= \tfrac{1}{2}|\hat{\mathbf{x}}64+\hat{\mathbf{y}}32+\hat{\mathbf{z}}8| = \tfrac{1}{2}\sqrt{64^2+32^2+8^2} = \tfrac{1}{2}\sqrt{5184} = 36,$$

.

Problem.4 Given $\mathbf{A} = \hat{\mathbf{x}}2-\hat{\mathbf{y}}3+\hat{\mathbf{z}}1$ and $\mathbf{B} = \hat{\mathbf{x}}B_x+\hat{\mathbf{y}}2+\hat{\mathbf{z}}B_z$:

(a) f nd B_x and B_z if **A** is parallel to **B**;
(b) f nd a relation between B_x and B_z if **A** is perpendicular to **B**.

Solution:

(a) If **A** is parallel to **B**, then their directions are equal or opposite: $\hat{\mathbf{a}}_A = \pm\hat{\mathbf{a}}_B$, or

$$\mathbf{A}/|\mathbf{A}| = \pm\mathbf{B}/|\mathbf{B}|,$$
$$\frac{\hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}}{\sqrt{14}} = \pm\frac{\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}B_z}{\sqrt{4 + B_x^2 + B_z^2}}.$$

From the y-component,

$$\frac{-3}{\sqrt{14}} = \frac{\pm 2}{\sqrt{4 + B_x^2 + B_z^2}}$$

If solved for the minus sign (which means that **A** and **B** must point in opposite directions for them to be parallel). Solving for $B_x^2 + B_z^2$,

$$B_x^2 + B_z^2 = \left(\frac{-2}{-3}\sqrt{14}\right)^2 - 4 = \frac{20}{9}.$$

From the x-component,

$$\frac{2}{\sqrt{14}} = \frac{-B_x}{\sqrt{56/9}}, \qquad B_x = \frac{-2\sqrt{56}}{3\sqrt{14}} = \frac{-4}{3}$$

and, from the z-component,

$$B_z = \frac{-2}{3}.$$

This is consistent with our result for $B_x^2 + B_z^2$.

These results could also have been obtained by assuming θ_{AB} was 0° or 180° and solving $|\mathbf{A}||\mathbf{B}| = \pm\mathbf{A}\cdot\mathbf{B}$, or by solving $\mathbf{A}\times\mathbf{B} = 0$.

(b) If **A** is perpendicular to **B**, then their dot product is zero.

,

$$0 = \mathbf{A}\cdot\mathbf{B} = 2B_x - 6 + B_z,$$

or

$$B_z = 6 - 2B_x.$$

There are an inf nite number of vectors which could be **B** and be perpendicular to **A**, but their x- and z-components must satisfy this relation.

This result could have also been obtained by assuming $\theta_{AB} = 90°$ and calculating $|\mathbf{A}||\mathbf{B}| = |\mathbf{A}\times\mathbf{B}|$.

Problem.5 Given vectors $\mathbf{A} = \hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3$, $\mathbf{B} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}4$, and $\mathbf{C} = \hat{\mathbf{y}}2 - \hat{\mathbf{z}}4$, f nd

(a) A and $\hat{\mathbf{a}}$,
(b) the component of $\mathbf{B}$ along $\mathbf{C}$,
(c) θ_{AC},
(d) $\mathbf{A} \times \mathbf{C}$,
(e) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$,
(f) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$,
(g) $\hat{\mathbf{x}} \times \mathbf{B}$, and
(h) $(\mathbf{A} \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{z}}$.

Solution:
(a)

$$A = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14},$$

and,

$$\hat{\mathbf{a}}_A = \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3}{\sqrt{14}} .$$

(b) The component of $\mathbf{B}$ along $\mathbf{C}$ is given by

$$B \cos\theta_{BC} = \frac{\mathbf{B} \cdot \mathbf{C}}{C} = \frac{-8}{\sqrt{20}} = -1.8 .$$

(c) From,

$$\theta_{AC} = \cos^{-1} \frac{\mathbf{A} \cdot \mathbf{C}}{AC} = \cos^{-1} \frac{4 + 12}{\sqrt{14}\sqrt{20}} = \cos^{-1} \frac{16}{\sqrt{280}} = 17.0°.$$

(d) From,

$$\mathbf{A} \times \mathbf{C} = \hat{\mathbf{x}}(2(-4) - (-3)2) + \hat{\mathbf{y}}((-3)0 - 1(-4)) + \hat{\mathbf{z}}(1(2) - 2(0)) = -\hat{\mathbf{x}}2 + \hat{\mathbf{y}}4 + \hat{\mathbf{z}}2.$$

(e) From,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \cdot (\hat{\mathbf{x}}16 + \hat{\mathbf{y}}8 + \hat{\mathbf{z}}4) = 1(16) + 2(8) + (-3)4 = 20.$$

.

(f)

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \times (\hat{\mathbf{x}}16 + \hat{\mathbf{y}}8 + \hat{\mathbf{z}}4) = \hat{\mathbf{x}}32 - \hat{\mathbf{y}}52 - \hat{\mathbf{z}}24.$$

.

(g) From,

$$\hat{\mathbf{x}} \times \mathbf{B} = -\hat{\mathbf{z}}4.$$

(h) From,

$$(\mathbf{A} \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{z}} = (\hat{\mathbf{x}}3 + \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} = 1.$$

Problem.6 Given vectors $\mathbf{A} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3$ and $\mathbf{B} = \hat{\mathbf{x}}3 - \hat{\mathbf{z}}2$, f nd a vector $\mathbf{C}$ whose magnitude is 9 and whose direction is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

Solution: The cross product of two vectors produces a new vector which is perpendicular to both of the original vectors. Two vectors exist which have a magnitude of 9 and are orthogonal to both $\mathbf{A}$ and $\mathbf{B}$: one which is 9 units long in the direction of the unit vector parallel to $\mathbf{A} \times \mathbf{B}$, and one in the opposite direction.

$$\begin{aligned} \mathbf{C} = \pm 9 \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} &= \pm 9 \frac{(\hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3) \times (\hat{\mathbf{x}}3 - \hat{\mathbf{z}}2)}{|(\hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3) \times (\hat{\mathbf{x}}3 - \hat{\mathbf{z}}2)|} \\ &= \pm 9 \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}13 + \hat{\mathbf{z}}3}{\sqrt{2^2 + 13^2 + 3^2}} \approx \pm(\hat{\mathbf{x}}1.34 + \hat{\mathbf{y}}8.67 + \hat{\mathbf{z}}2.0). \end{aligned}$$

Problem.7 Given $\mathbf{A} = \hat{\mathbf{x}}(x+2y) - \hat{\mathbf{y}}(y+3z) + \hat{\mathbf{z}}(3x-y)$, determine a unit vector parallel to $\mathbf{A}$ at point $P(1,-1,2)$.

Solution: The unit vector parallel to $\mathbf{A} = \hat{\mathbf{x}}(x+2y) - \hat{\mathbf{y}}(y+3z) + \hat{\mathbf{z}}(3x-y)$ at the point $P(1,-1,2)$ is

$$\frac{\mathbf{A}(1,-1,2)}{|\mathbf{A}(1,-1,2)|} = \frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}}5 + \hat{\mathbf{z}}4}{\sqrt{(-1)^2 + (-5)^2 + 4^2}} = \frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}}5 + \hat{\mathbf{z}}4}{\sqrt{42}} \approx -\hat{\mathbf{x}}0.15 - \hat{\mathbf{y}}0.77 + \hat{\mathbf{z}}0.62.$$

Problem.8 By expansion in Cartesian coordinates, prove:
(a) the relation for the scalar triple product, and
(b) the relation for the vector triple product.

Solution:
(a) Proof of the scalar triple product given by,

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) + \hat{\mathbf{z}}(A_xB_y - A_yB_x),$$

$$\mathbf{B}\times\mathbf{C} = \hat{\mathbf{x}}(B_yC_z - B_zC_y) + \hat{\mathbf{y}}(B_zC_x - B_xC_z) + \hat{\mathbf{z}}(B_xC_y - B_yC_x),$$
$$\mathbf{C}\times\mathbf{A} = \hat{\mathbf{x}}(C_yA_z - C_zA_y) + \hat{\mathbf{y}}(C_zA_x - C_xA_z) + \hat{\mathbf{z}}(C_xA_y - C_yA_x).$$

it is easily shown that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x),$$
$$\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = B_x(C_yA_z - C_zA_y) + B_y(C_zA_x - C_xA_z) + B_z(C_xA_y - C_yA_x),$$
$$\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = C_x(A_yB_z - A_zB_y) + C_y(A_zB_x - A_xB_z) + C_z(A_xB_y - A_yB_x),$$

which are all the same.

(b) The evaluation of the left hand side employs the expression above for $\mathbf{B}\times\mathbf{C}$:

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{A}\times(\hat{\mathbf{x}}(B_yC_z - B_zC_y) + \hat{\mathbf{y}}(B_zC_x - B_xC_z) + \hat{\mathbf{z}}(B_xC_y - B_yC_x))\\ &= \hat{\mathbf{x}}(A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z))\\ &\quad + \hat{\mathbf{y}}(A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x))\\ &\quad + \hat{\mathbf{z}}(A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)),\end{aligned}$$

while the right hand side is

$$\begin{aligned}\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{B}(A_xC_x + A_yC_y + A_zC_z) - \mathbf{C}(A_xB_x + A_yB_y + A_zB_z)\\ &= \hat{\mathbf{x}}(B_x(A_yC_y + A_zC_z) - C_x(A_yB_y + A_zB_z))\\ &\quad + \hat{\mathbf{y}}(B_y(A_xC_x + A_zC_z) - C_y(A_xB_x + A_zB_z))\\ &\quad + \hat{\mathbf{z}}(B_z(A_xC_x + A_yC_y) - C_z(A_xB_x + A_yB_y)).\end{aligned}$$

By rearranging the expressions for the components, the left hand side is equal to the right hand side.

Problem.9 Find an expression for the unit vector directed toward the origin from an arbitrary point on the line described by $x = 1$ and $z = 2$.

Solution: An arbitrary point on the given line is $(1, y, 2)$. The vector from this point to $(0, 0, 0)$ is:

$$\mathbf{A} = \hat{\mathbf{x}}(0-1) + \hat{\mathbf{y}}(0-y) + \hat{\mathbf{z}}(0-2) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}y - 2\hat{\mathbf{z}},$$
$$|\mathbf{A}| = \sqrt{1 + y^2 + 4} = \sqrt{5 + y^2},$$
$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}}y - \hat{\mathbf{z}}2}{\sqrt{5 + y^2}}.$$

Problem.10 Find an expression for the unit vector directed toward the point P located on the z-axis at a height h above the x–y plane from an arbitrary point $Q(x,y,-3)$ in the plane $z=-3$.

Solution: Point P is at $(0,0,h)$. Vector **A** from $Q(x,y,-3)$ to $P(0,0,h)$ is:

$$\mathbf{A} = \hat{\mathbf{x}}(0-x)+\hat{\mathbf{y}}(0-y)+\hat{\mathbf{z}}(h+3) = -\hat{\mathbf{x}}x-\hat{\mathbf{y}}y+\hat{\mathbf{z}}(h+3),$$

$$|\mathbf{A}| = [x^2+y^2+(h+3)^2]^{1/2},$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{-\hat{\mathbf{x}}x-\hat{\mathbf{y}}y+\hat{\mathbf{z}}(h+3)}{[x^2+y^2+(h+3)^2]^{1/2}}.$$

Problem.11 Find a unit vector parallel to either direction of the line described by

$$2x+z=4.$$

Solution: First, we f nd any two points on the given line. Since the line equation is not a function of y, the given line is in a plane parallel to the x–z plane. For convenience, we choose the x–z plane with $y=0$.

For $x=0$, $z=4$. Hence, point P is at $(0,0,4)$.

For $z=0$, $x=2$. Hence, point Q is at $(2,0,0)$.

Vector **A** from P to Q is:

$$\mathbf{A} = \hat{\mathbf{x}}(2-0)+\hat{\mathbf{y}}(0-0)+\hat{\mathbf{z}}(0-4) = \hat{\mathbf{x}}2-\hat{\mathbf{z}}4,$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}}2-\hat{\mathbf{z}}4}{\sqrt{20}}.$$

Problem.12 Two lines in the x–y plane are described by the expressions:

Line 1	$x+2y=-6$,
Line 2	$3x+4y=8$.

Use vector algebra to f nd the smaller angle between the lines at their intersection point.

Solution: Intersection point is found by solving the two equations simultaneously:

$$-2x-4y=12,$$

$$3x+4y=8.$$

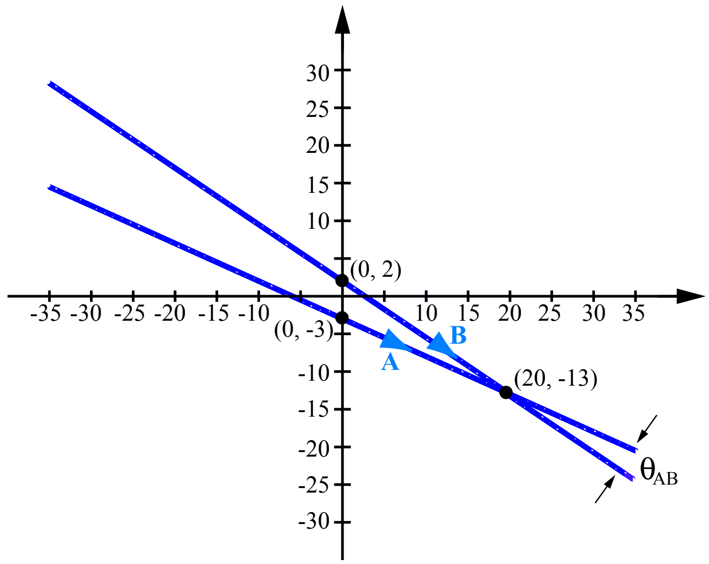


Figure P.12: Lines 1 and 2.

The sum gives $x = 20$, which, when used in the f rst equation, gives $y = -13$.

Hence, intersection point is $(20, -13)$.

Another point on line 1 is $x = 0,\ y = -3$. Vector **A** from $(0, -3)$ to $(20, -13)$ is

$$\mathbf{A} = \hat{\mathbf{x}}(20) + \hat{\mathbf{y}}(-13+3) = \hat{\mathbf{x}}20 - \hat{\mathbf{y}}10,$$
$$|\mathbf{A}| = \sqrt{20^2 + 10^2} = \sqrt{500}.$$

A point on line 2 is $x = 0,\ y = 2$. Vector **B** from $(0, 2)$ to $(20, -13)$ is

$$\mathbf{B} = \hat{\mathbf{x}}(20) + \hat{\mathbf{y}}(-13-2) = \hat{\mathbf{x}}20 - \hat{\mathbf{y}}15,$$
$$|\mathbf{B}| = \sqrt{20^2 + 15^2} = \sqrt{625}.$$

Angle between **A** and **B** is

$$\theta_{AB} = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right) = \cos^{-1}\left(\frac{400+150}{\sqrt{500}\cdot\sqrt{625}}\right) = 10.3°.$$

Problem.13 A given line is described by

$$x + 2y = 4.$$

Vector **A** starts at the origin and ends at point P on the line such that **A** is orthogonal to the line. Find an expression for **A**.

Solution: We f rst plot the given line. Next we f nd vector **B** which connects point $P_1(0,2)$ to $P_2(4,0)$, both of which are on the line:

$$\mathbf{B} = \hat{\mathbf{x}}(4-0) + \hat{\mathbf{y}}(0-2) = \hat{\mathbf{x}}4 - \hat{\mathbf{y}}2.$$

Vector **A** starts at the origin and ends on the line at P. If the x-coordinate of P is x,

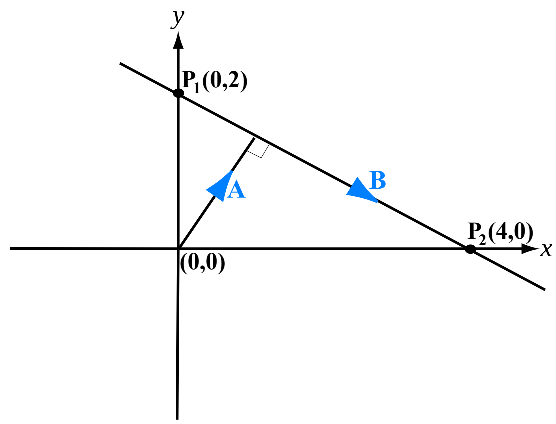


Figure P.13: Given line and vector **A**.

then its y-coordinate has to be $(4-x)/2$ in order to be on the line. Hence P is at $(x,\,(4-x)/2)$. Vector **A** is

$$\mathbf{A} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}\left(\frac{4-x}{2}\right).$$

But **A** is perpendicular to the line. Hence,

$$\mathbf{A}\cdot\mathbf{B} = 0,$$

$$\left[\hat{\mathbf{x}}x + \hat{\mathbf{y}}\left(\frac{4-x}{2}\right)\right]\cdot(\hat{\mathbf{x}}4 - \hat{\mathbf{y}}2) = 0,$$

$$4x - (4-x) = 0, \quad \text{or}$$

$$x = \frac{4}{5} = 0.8.$$

Hence,

$$\mathbf{A} = \hat{\mathbf{x}}0.8 + \hat{\mathbf{y}}\left(\frac{4-0.8}{2}\right) = \hat{\mathbf{x}}0.8 + \hat{\mathbf{y}}1.6.$$

Problem.14 A certain plane is described by

$$2x + 3y + 4z = 16.$$

Find the unit vector normal to the surface in the direction away from the origin.

Solution: Procedure:

1. Use the equation for the given plane to find three points, P_1, P_2 and P_3 on the plane.

2. Find vector **A** from P_1 to P_2 and vector **B** from P_1 to P_3.

3. Cross product of **A** and **B** gives a vector **C** orthogonal to **A** and **B**, and hence to the plane.

4. Check direction of $\hat{\mathbf{c}}$.

Steps:

1. Choose the following three points:

$$\begin{aligned}&P_1 \text{ at } (0,0,4),\\ &P_2 \text{ at } (8,0,0),\\ &P_3 \text{ at } (0,\tfrac{16}{3},0).\end{aligned}$$

2. Vector **A** from P_1 to P_2

$$\mathbf{A} = \hat{\mathbf{x}}\,(8-0) + \hat{\mathbf{y}}\,(0-0) + \hat{\mathbf{z}}\,(0-4) = \hat{\mathbf{x}}\,8 - \hat{\mathbf{z}}\,4$$

Vector **B** from P_1 to P_3

$$\mathbf{B} = \hat{\mathbf{x}}\,(0-0) + \hat{\mathbf{y}}\left(\frac{16}{3} - 0\right) + \hat{\mathbf{z}}\,(0-4) = \hat{\mathbf{y}}\,\frac{16}{3} - \hat{\mathbf{z}}\,4$$

3.

$$\begin{aligned}\mathbf{C} &= \mathbf{A} \times \mathbf{B}\\ &= \hat{\mathbf{x}}\,(A_yB_z - A_zB_y) + \hat{\mathbf{y}}\,(A_zB_x - A_xB_z) + \hat{\mathbf{z}}\,(A_xB_y - A_yB_x)\\ &= \hat{\mathbf{x}}\left(0\cdot(-4) - (-4)\cdot\frac{16}{3}\right) + \hat{\mathbf{y}}\,((-4)\cdot 0 - 8\cdot(-4)) + \hat{\mathbf{z}}\left(8\cdot\frac{16}{3} - 0\cdot 0\right)\\ &= \hat{\mathbf{x}}\,\frac{64}{3} + \hat{\mathbf{y}}\,32 + \hat{\mathbf{z}}\,\frac{128}{3}\end{aligned}$$

Verify that **C** is orthogonal to **A** and **B**

$$\mathbf{A}\cdot\mathbf{C} = \left(8\cdot\frac{64}{3}\right) + (32\cdot 0) + \left(\frac{128}{3}\cdot(-4)\right) = \frac{512}{3} - \frac{512}{3} = 0$$
$$\mathbf{B}\cdot\mathbf{C} = \left(0\cdot\frac{64}{3}\right) + \left(32\cdot\frac{16}{3}\right) + \left(\frac{128}{3}\cdot(-4)\right) = \frac{512}{3} - \frac{512}{3} = 0$$

4. $\mathbf{C} = \hat{\mathbf{x}}\frac{64}{3} + \hat{\mathbf{y}}32 + \hat{\mathbf{z}}\frac{128}{3}$

$$\hat{\mathbf{c}} = \frac{\mathbf{C}}{|\mathbf{C}|} = \frac{\hat{\mathbf{x}}\frac{64}{3} + \hat{\mathbf{y}}32 + \hat{\mathbf{z}}\frac{128}{3}}{\sqrt{\left(\frac{64}{3}\right)^2 + 32^2 + \left(\frac{128}{3}\right)^2}} = \hat{\mathbf{x}}0.37 + \hat{\mathbf{y}}0.56 + \hat{\mathbf{z}}0.74.$$

$\hat{\mathbf{c}}$ points away from the origin as desired.

Problem.15 Given $\mathbf{B} = \hat{\mathbf{x}}(z-3y) + \hat{\mathbf{y}}(2x-3z) - \hat{\mathbf{z}}(x+y)$, find a unit vector parallel to **B** at point $P(1,0,-1)$.

Solution: At $P(1,0,-1)$,

$$\mathbf{B} = \hat{\mathbf{x}}(-1) + \hat{\mathbf{y}}(2+3) - \hat{\mathbf{z}}(1) = -\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}},$$
$$\hat{\mathbf{b}} = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}}{\sqrt{1+25+1}} = \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}}{\sqrt{27}}.$$

Problem.16 = Using arrow representation, sketch each of the following vector fields:

(a) $\mathbf{E}_1 = -\hat{\mathbf{x}}y$,

(b) $\mathbf{E}_2 = \hat{\mathbf{y}}x$,

(c) $\mathbf{E}_3 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$,

(d) $\mathbf{E}_4 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}2y$,

(e) $\mathbf{E}_5 = \hat{\boldsymbol{\phi}}r$,

(f) $\mathbf{E}_6 = \hat{\mathbf{r}}\sin\phi$.

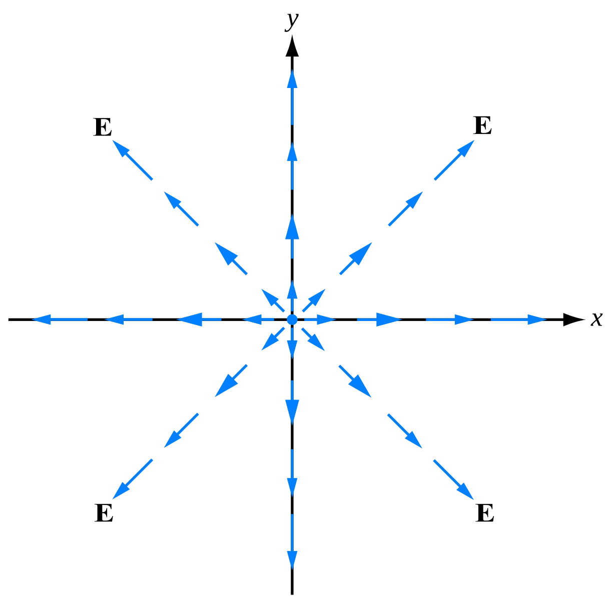
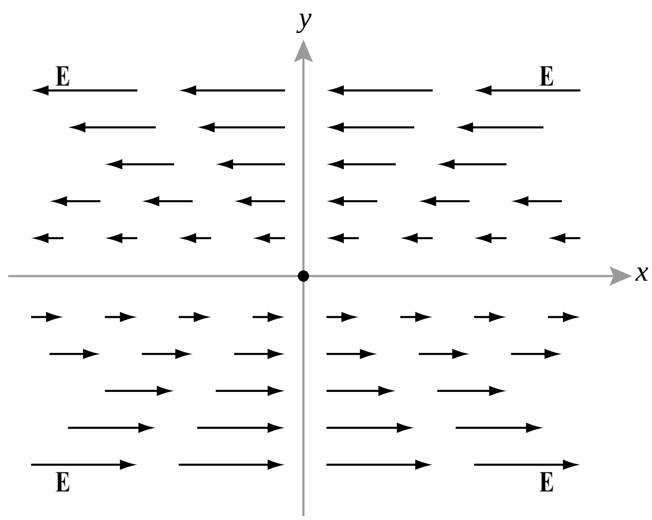


Figure P.16: Arrow representation for vector f eld $\mathbf{E} = \hat{\mathbf{r}}r$

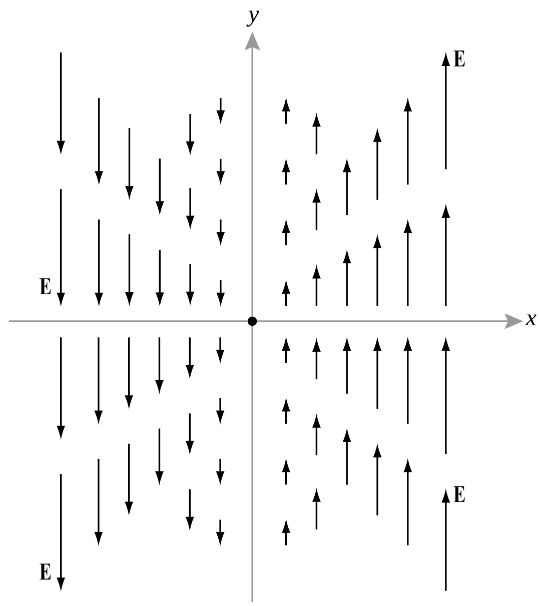
Solution:

(a)



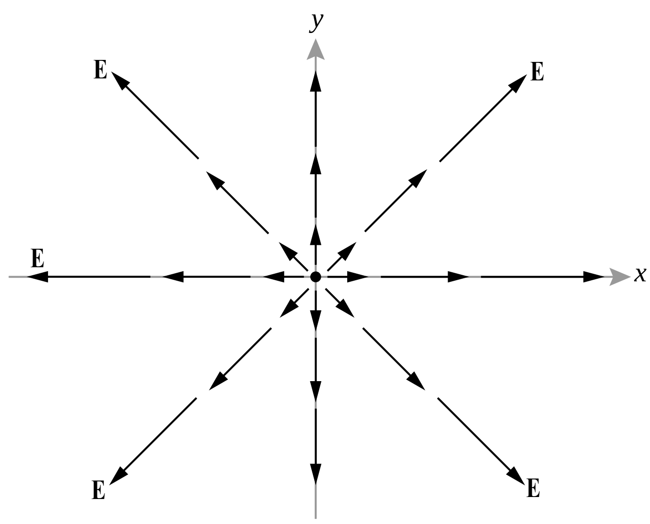
$\mathbf{E}_1 = -\hat{\mathbf{x}}\, y$

(b)



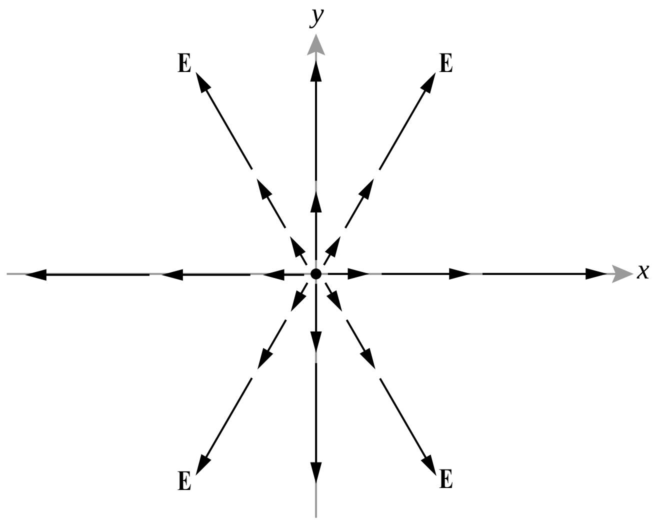
P $\mathbf{E}_2 = \quad \hat{\mathbf{y}} x$

(c)



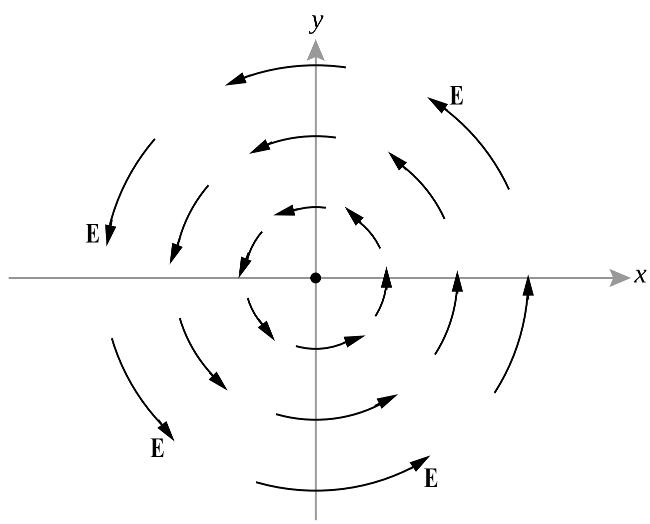
$\mathbf{E}_3 = \hat{\mathbf{x}}\,x + \hat{\mathbf{y}}\,y$

(d)



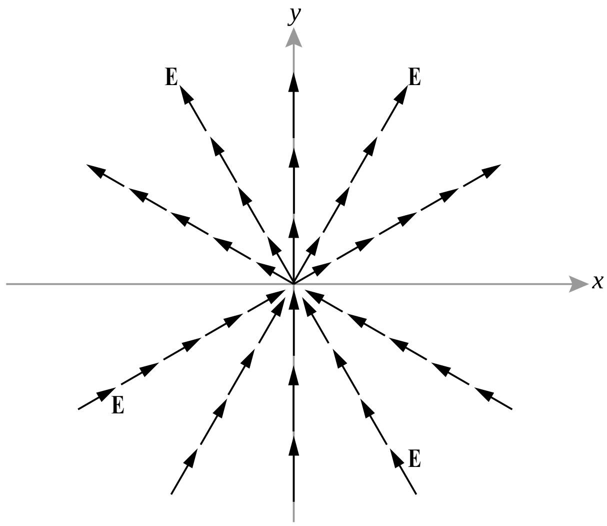
$$\mathbf{E}_4 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}2y$$

(e)



$$\mathbf{E}_5 = \hat{\boldsymbol{\phi}}r$$

(f)



$E_6 = \hat{\mathbf{r}} \sin\phi$

Problem.17 Use arrows to sketch each of the following vector f elds:

(a) $\mathbf{E}_1 = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$,

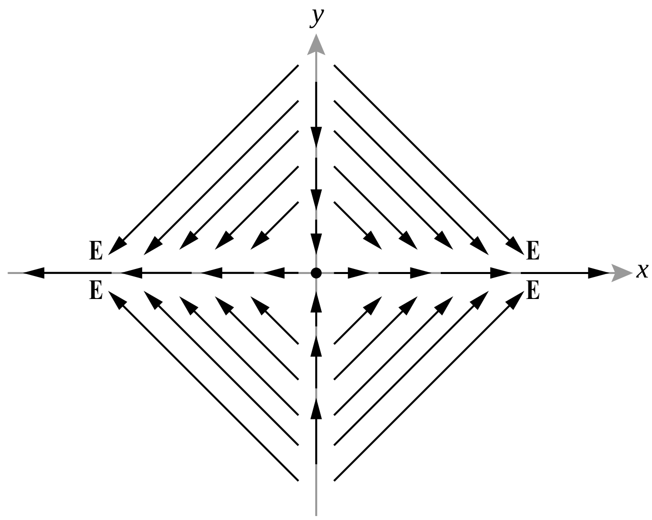
(b) $\mathbf{E}_2 = -\hat{\boldsymbol{\phi}}$,

(c) $\mathbf{E}_3 = \hat{\mathbf{y}}\frac{1}{x}$,

(d) $\mathbf{E}_4 = \hat{\mathbf{r}}\cos\phi$.

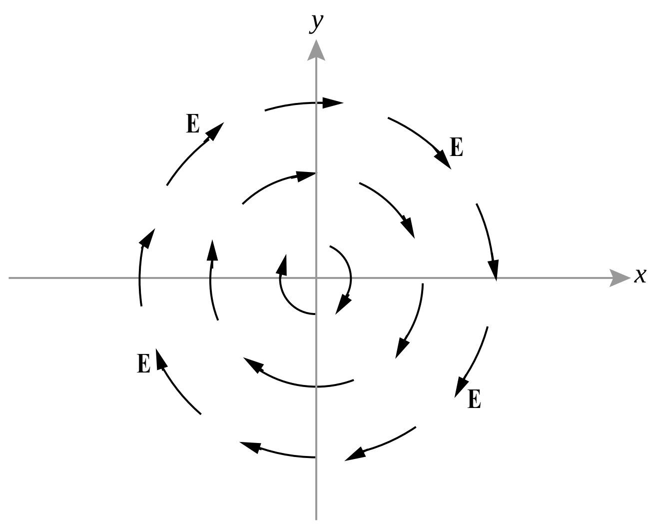
Solution:

(a)



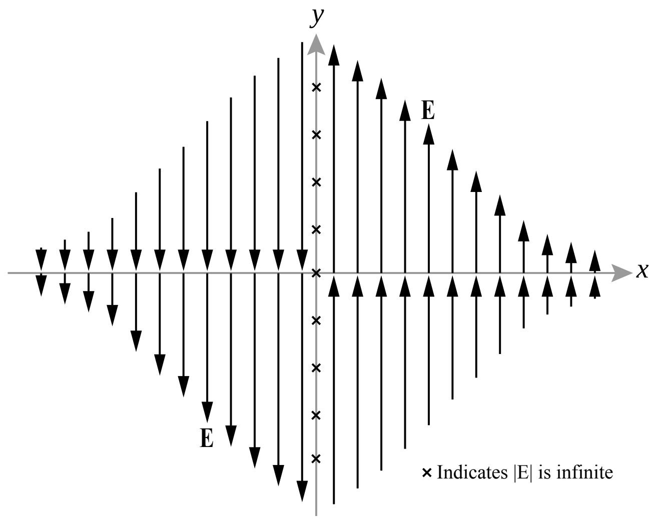
$E_1 = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$

(b)



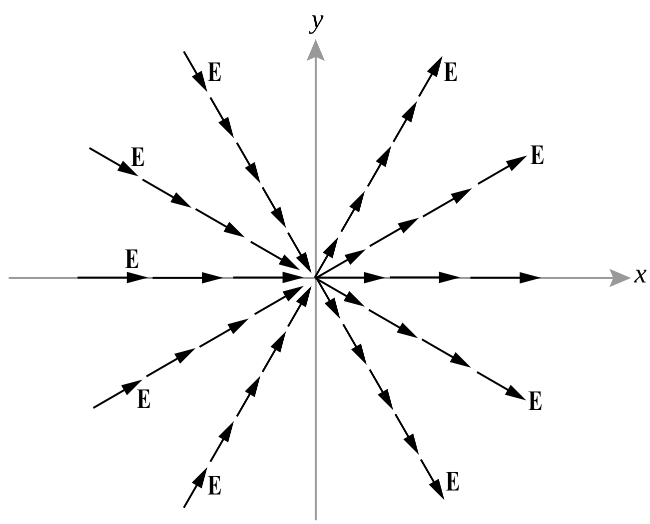
$E_2 = -\hat{\boldsymbol{\phi}}$

(c)



$\mathbf{E}_3 = \hat{\mathbf{y}}\,(1/x)$

(d)



$\mathbf{E}_4 = \hat{\mathbf{r}}\cos\phi$

Sections-2: Coordinate Systems

Problem.18 Convert the coordinates of the following points from Cartesian to cylindrical and spherical coordinates:

(a) $P_1(1,2,0)$,
(b) $P_2(0,0,2)$,
(c) $P_3(1,1,3)$,
(d) $P_4(-2,2,-2)$.

Solution: Use the "coordinate variables" column in the Table.

(a) In the cylindrical coordinate system,

$$P_1 = (\sqrt{1^2+2^2}, \tan^{-1}(2/1), 0) = (\sqrt{5}, 1.107 \text{ rad}, 0) \approx (2.24, 63.4^\circ, 0).$$

In the spherical coordinate system,

$$\begin{aligned} P_1 &= (\sqrt{1^2+2^2+0^2}, \tan^{-1}(\sqrt{1^2+2^2}/0), \tan^{-1}(2/1)) \\ &= (\sqrt{5}, \pi/2 \text{ rad}, 1.107 \text{ rad}) \approx (2.24, 90.0^\circ, 63.4^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant I.

(b) In the cylindrical coordinate system,

$$P_2 = (\sqrt{0^2+0^2}, \tan^{-1}(0/0), 2) = (0, 0 \text{ rad}, 2) = (0, 0^\circ, 2).$$

In the spherical coordinate system,

$$\begin{aligned} P_2 &= (\sqrt{0^2+0^2+2^2}, \tan^{-1}(\sqrt{0^2+0^2}/2), \tan^{-1}(0/0)) \\ &= (2, 0 \text{ rad}, 0 \text{ rad}) = (2, 0^\circ, 0^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is arbitrary and may take any value.

(c) In the cylindrical coordinate system,

$$P_3 = (\sqrt{1^2+1^2}, \tan^{-1}(1/1), 3) = (\sqrt{2}, \pi/4 \text{ rad}, 3) \approx (1.41, 45.0^\circ, 3).$$

In the spherical coordinate system,

$$\begin{aligned} P_3 &= (\sqrt{1^2+1^2+3^2}, \tan^{-1}(\sqrt{1^2+1^2}/3), \tan^{-1}(1/1)) \\ &= (\sqrt{11}, 0.44 \text{ rad}, \pi/4 \text{ rad}) \approx (3.32, 25.2^\circ, 45.0^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant I.

(d) In the cylindrical coordinate system,

$$\begin{aligned}P_4 &= (\sqrt{(-2)^2+2^2}, \tan^{-1}(2/-2), -2)\\ &= (2\sqrt{2}, 3\pi/4 \text{ rad}, -2) \approx (2.83, 135.0^\circ, -2).\end{aligned}$$

In the spherical coordinate system,

$$\begin{aligned}P_4 &= (\sqrt{(-2)^2+2^2+(-2)^2}, \tan^{-1}(\sqrt{(-2)^2+2^2}/-2), \tan^{-1}(2/-2))\\ &= (2\sqrt{3}, 2.187 \text{ rad}, 3\pi/4 \text{ rad}) \approx (3.46, 125.3^\circ, 135.0^\circ).\end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant II.

Problem.19 Convert the coordinates of the following points from cylindrical to Cartesian coordinates:

(a) $P_1(2, \pi/4, -2)$,
(b) $P_2(3, 0, -2)$,
(c) $P_3(4, \pi, 3)$.

Solution:
(a)

$$P_1(x,y,z) = P_1(r\cos\phi, r\sin\phi, z) = P_1\left(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}, -2\right) = P_1(1.41, 1.41, -2).$$

(b) $P_2(x,y,z) = P_2(3\cos 0, 3\sin 0, -2) = P_2(3, 0, -2)$.
(c) $P_3(x,y,z) = P_3(4\cos\pi, 4\sin\pi, 3) = P_3(-4, 0, 3)$.

Problem.20 Convert the coordinates of the following points from spherical to cylindrical coordinates:

(a) $P_1(5, 0, 0)$,
(b) $P_2(5, 0, \pi)$,
(c) $P_3(3, \pi/2, 0)$.

Solution:
(a)

$$\begin{aligned}P_1(r,\phi,z) = P_1(R\sin\theta, \phi, R\cos\theta) &= P_1(5\sin 0, 0, 5\cos 0)\\ &= P_1(0, 0, 5).\end{aligned}$$

(b) $P_2(r,\phi,z) = P_2(5\sin 0, \pi, 5\cos 0) = P_2(0, \pi, 5)$.

(c) $P_3(r,\phi,z) = P_3(3\sin\frac{\pi}{2}, 0, 3\cos\frac{\pi}{2}) = P_3(3,0,0)$.

Problem.21 Use the appropriate expression for the differential surface area $d\mathbf{s}$ to determine the area of each of the following surfaces:

(a) $r = 3;\ 0 \le \phi \le \pi/3;\ -2 \le z \le 2$,

(b) $2 \le r \le 5;\ \pi/2 \le \phi \le \pi;\ z = 0$,

(c) $2 \le r \le 5;\ \phi = \pi/4;\ -2 \le z \le 2$,

(d) $R = 2;\ 0 \le \theta \le \pi/3;\ 0 \le \phi \le \pi$,

(e) $0 \le R \le 5;\ \theta = \pi/3;\ 0 \le \phi \le 2\pi$.

Also sketch the outlines of each of the surfaces.

Solution:

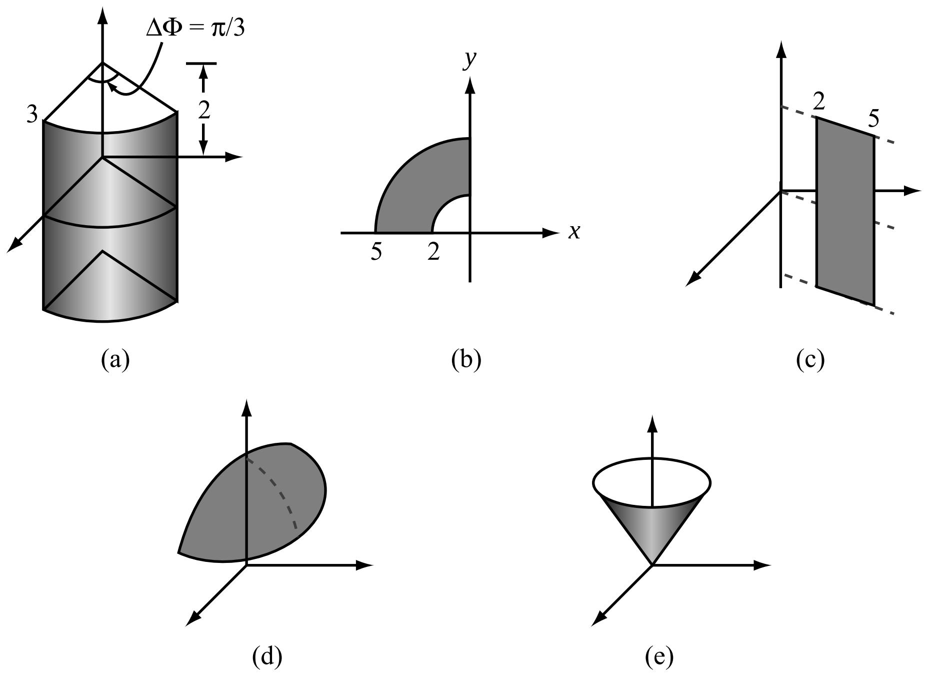


Figure P.21: Surfaces described by Problem.21.

(a) Using,

$$A = \int_{z=-2}^{2} \int_{\phi=0}^{\pi/3} (r)|_{r=3}\, d\phi\, dz = \left((3\phi z)\big|_{\phi=0}^{\pi/3} \right)\Big|_{z=-2}^{2} = 4\pi.$$

(b) Using,

$$A = \int_{r=2}^{5} \int_{\phi=\pi/2}^{\pi} (r)|_{z=0}\, d\phi\, dr = \left(\left(\tfrac{1}{2} r^2 \phi\right)\Big|_{r=2}^{5} \right)\Big|_{\phi=\pi/2}^{\pi} = \frac{21\pi}{4}.$$

(c) Using,

$$A = \int_{z=-2}^{2} \int_{r=2}^{5} (1)|_{\phi=\pi/4}\, dr\, dz = \left((rz)\Big|_{z=-2}^{2} \right)\Big|_{r=2}^{5} = 12.$$

(d) Using,

$$A = \int_{\theta=0}^{\pi/3} \int_{\phi=0}^{\pi} \left(R^2 \sin\theta\right)\Big|_{R=2} d\phi\, d\theta = \left((-4\phi\cos\theta)\Big|_{\theta=0}^{\pi/3} \right)\Big|_{\phi=0}^{\pi} = 2\pi.$$

(e) Using,

$$A = \int_{R=0}^{5} \int_{\phi=0}^{2\pi} (R\sin\theta)|_{\theta=\pi/3}\, d\phi\, dR = \left(\left(\tfrac{1}{2} R^2 \phi \sin\frac{\pi}{3}\right)\Big|_{\phi=0}^{2\pi} \right)\Big|_{R=0}^{5} = \frac{25\sqrt{3}\pi}{2}.$$

Problem.22 Find the volumes described by

(a) $2 \le r \le 5;\ \pi/2 \le \phi \le \pi;\ 0 \le z \le 2,$

(b) $0 \le R \le 5;\ 0 \le \theta \le \pi/3;\ 0 \le \phi \le 2\pi.$

Also sketch the outline of each volume.

Solution:

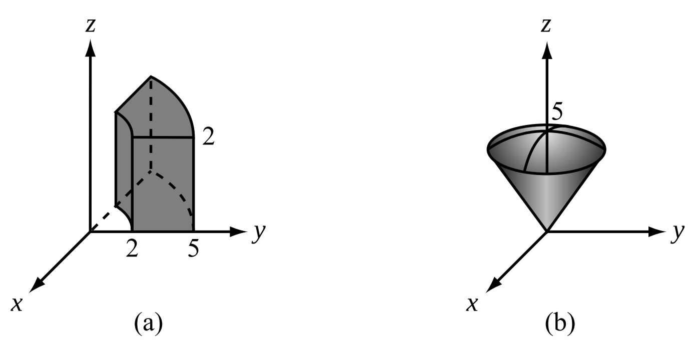


Figure P.22: Volumes described by Problem.22 .

(a) From,

$$V = \int_{z=0}^{2} \int_{\phi=\pi/2}^{\pi} \int_{r=2}^{5} r\, dr\, d\phi\, dz = \left(\left(\left(\tfrac{1}{2} r^2 \phi z\right)\Big|_{r=2}^{5} \right)\Big|_{\phi=\pi/2}^{\pi} \right)\Big|_{z=0}^{2} = \frac{21\pi}{2}.$$

(b) From,

$$V = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/3} \int_{R=0}^{5} R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= \left(\left(\left(-\cos\theta \frac{R^3}{3} \phi \right) \Big|_{R=0}^{5} \right) \Big|_{\theta=0}^{\pi/3} \right) \Big|_{\phi=0}^{2\pi} = \frac{125\pi}{3}.$$

Problem.23 A section of a sphere is described by $0 \leq R \leq 2$, $0 \leq \theta \leq 90°$, and $30° \leq \phi \leq 90°$. Find:

(a) the surface area of the spherical section,

(b) the enclosed volume.

Also sketch the outline of the section.

Solution:

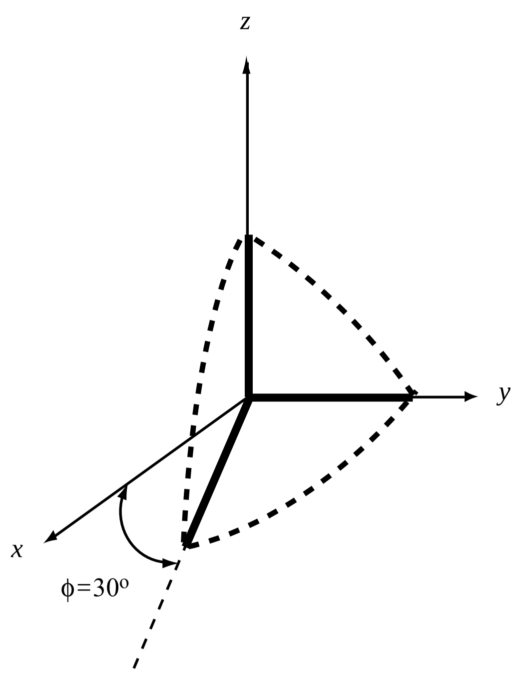


Figure P.23: Outline of section.

$$S = \int_{\phi=\pi/6}^{\pi/2} \int_{\theta=0}^{\pi/2} R^2 \sin\theta \, d\theta \, d\phi |_{R=2}$$

$$= 4\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \left[-\cos\theta|_0^{\pi/2}\right] = 4 \times \frac{\pi}{3} = \frac{4\pi}{3} \quad (\text{m}^2),$$

$$V = \int_{R=0}^{2} \int_{\phi=\pi/6}^{\pi/2} \int_{\theta=0}^{\pi/2} R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= \left.\frac{R^3}{3}\right|_0^2 \left(\frac{\pi}{2} - \frac{\pi}{6}\right) [-\cos\theta|_0^{\pi/2}] = \frac{8}{3} \frac{\pi}{3} = \frac{8\pi}{9} \quad (\text{m}^3).$$

Problem.24 A vector f eld is given in cylindrical coordinates by

$$\mathbf{E} = \hat{\mathbf{r}} r \cos\phi + \hat{\boldsymbol{\phi}} r \sin\phi + \hat{\mathbf{z}} z^2.$$

Point $P(2, \pi, 3)$ is located on the surface of the cylinder described by $r = 2$. At point P, f nd:

(a) the vector component of **E** perpendicular to the cylinder,

(b) the vector component of **E** tangential to the cylinder.

Solution:

(a) $\mathbf{E}_\text{n} = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{E}) = \hat{\mathbf{r}}[\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}} r \cos\phi + \hat{\boldsymbol{\phi}} r \sin\phi + \hat{\mathbf{z}} z^2)] = \hat{\mathbf{r}} r \cos\phi$.
At $P(2, \pi, 3)$, $\mathbf{E}_\text{n} = \hat{\mathbf{r}} 2 \cos\pi = -\hat{\mathbf{r}} 2$.
(b) $\mathbf{E}_\text{t} = \mathbf{E} - \mathbf{E}_\text{n} = \hat{\boldsymbol{\phi}} r \sin\phi + \hat{\mathbf{z}} z^2$.
At $P(2, \pi, 3)$, $\mathbf{E}_\text{t} = \hat{\boldsymbol{\phi}} 2 \sin\pi + \hat{\mathbf{z}} 3^2 = \hat{\mathbf{z}} 9$.

Problem.25 At a given point in space, vectors **A** and **B** are given in spherical coordinates by

$$\mathbf{A} = \hat{\mathbf{R}} 4 + \hat{\boldsymbol{\theta}} 2 - \hat{\boldsymbol{\phi}},$$
$$\mathbf{B} = -\hat{\mathbf{R}} 2 + \hat{\boldsymbol{\phi}} 3.$$

Find:

(a) the scalar component, or projection, of **B** in the direction of **A**,

(b) the vector component of **B** in the direction of **A**,

(c) the vector component of **B** perpendicular to **A**.

Solution:

(a) Scalar component of **B** in direction of **A**:

$$C = \mathbf{B} \cdot \hat{\mathbf{a}} = \mathbf{B} \cdot \frac{\mathbf{A}}{|\mathbf{A}|} = (-\hat{\mathbf{R}}2 + \hat{\boldsymbol{\phi}}3) \cdot \frac{(\hat{\mathbf{R}}4 + \hat{\boldsymbol{\theta}}2 - \hat{\boldsymbol{\phi}})}{\sqrt{16+4+1}}$$
$$= \frac{-8-3}{\sqrt{21}} = -\frac{11}{\sqrt{21}} = -2.4.$$

(b) Vector component of **B** in direction of **A**:

$$\mathbf{C} = \hat{\mathbf{a}}C = \mathbf{A}\frac{C}{|\mathbf{A}|} = (\hat{\mathbf{R}}4 + \hat{\boldsymbol{\theta}}2 - \hat{\boldsymbol{\phi}})\,\frac{(-2.4)}{\sqrt{21}}$$
$$= -(\hat{\mathbf{R}}2.09 + \hat{\boldsymbol{\theta}}1.05 - \hat{\boldsymbol{\phi}}0.52).$$

(c) Vector component of **B** perpendicular to **A**:

$$\mathbf{D} = \mathbf{B} - \mathbf{C} = (-\hat{\mathbf{R}}2 + \hat{\boldsymbol{\phi}}3) + (\hat{\mathbf{R}}2.09 + \hat{\boldsymbol{\theta}}1.05 - \hat{\boldsymbol{\phi}}0.52)$$
$$= \hat{\mathbf{R}}0.09 + \hat{\boldsymbol{\theta}}1.05 + \hat{\boldsymbol{\phi}}2.48.$$

Problem.26 Given vectors

$$\mathbf{A} = \hat{\mathbf{r}}(\cos\phi + 3z) - \hat{\boldsymbol{\phi}}(2r + 4\sin\phi) + \hat{\mathbf{z}}(r - 2z),$$
$$\mathbf{B} = -\hat{\mathbf{r}}\sin\phi + \hat{\mathbf{z}}\cos\phi,$$

f nd

(a) θ_{AB} at $(2, \pi/2, 0)$,

(b) a unit vector perpendicular to both **A** and **B** at $(2, \pi/3, 1)$.

Solution: It doesn't matter whether the vectors are evaluated before vector products are calculated, or if the vector products are directly calculated and the general results are evaluated at the specif c point in question.

(a) At $(2, \pi/2, 0)$, $\mathbf{A} = -\hat{\boldsymbol{\phi}}8 + \hat{\mathbf{z}}2$ and $\mathbf{B} = -\hat{\mathbf{r}}$. From Eq. (3.21),

$$\theta_{AB} = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) = \cos^{-1}\left(\frac{0}{AB}\right) = 90°.$$

(b) At $(2, \pi/3, 1)$, $\mathbf{A} = \hat{\mathbf{r}}\frac{7}{2} - \hat{\boldsymbol{\phi}}4(1 + \frac{1}{2}\sqrt{3})$ and $\mathbf{B} = -\hat{\mathbf{r}}\frac{1}{2}\sqrt{3} + \hat{\mathbf{z}}\frac{1}{2}$. Since $\mathbf{A} \times \mathbf{B}$ is perpendicular to both **A** and **B**, a unit vector perpendicular to both **A** and **B** is given by

$$\pm\frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} = \pm\frac{\hat{\mathbf{r}}(-4(1+\frac{1}{2}\sqrt{3}))(\frac{1}{2}) - \hat{\boldsymbol{\phi}}(\frac{7}{2})(\frac{1}{2}) - \hat{\mathbf{z}}(4(1+\frac{1}{2}\sqrt{3}))(\frac{1}{2}\sqrt{3})}{\sqrt{(2(1+\frac{1}{2}\sqrt{3}))^2 + (\frac{7}{4})^2 + (3+2\sqrt{3})^2}}$$
$$\approx \mp(\hat{\mathbf{r}}0.487 + \hat{\boldsymbol{\phi}}0.228 + \hat{\mathbf{z}}0.843).$$