Tutorial#2

Chapter2

Sections-1: Charge and Current Distributions

Problem.1 A cube 2 m on a side is located in the first octant in a Cartesian coordinate system, with one of its corners at the origin. Find the total charge contained in the cube if the charge density is given by $\rho_v = xy^2 e^{-2z}$ (mC/m^3).

Solution: For the cube shown in Fig. P.1,

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{2} xy^2 e^{-2z} \, dx \, dy \, dz$$

$$= \left(\frac{-1}{12} x^2 y^3 e^{-2z} \right) \Bigg|_{x=0}^{2} \Bigg|_{y=0}^{2} \Bigg|_{z=0}^{2} = \frac{8}{3}(1 - e^{-4}) = 2.62 \text{ mC}.$$

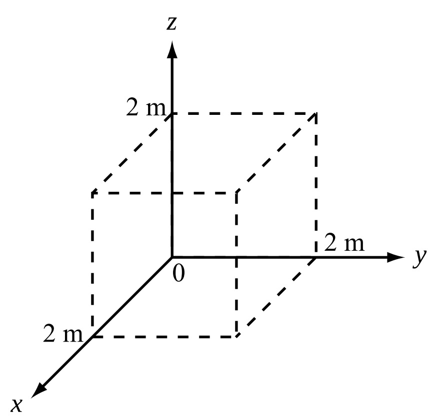

Figure P.1: Cube of Problem.1

Problem.2 Find the total charge contained in a cylindrical volume defined by $r \leq 2$ m and $0 \leq z \leq 3$ m if $\rho_v = 20rz$ (mC/m^3).

Solution: For the cylinder shown in Fig. P.2,

$$Q = \int_{z=0}^{3} \int_{\phi=0}^{2\pi} \int_{r=0}^{2} 20rz \, r \, dr \, d\phi \, dz$$

$$= \left(\frac{10}{3} r^3 \phi z^2 \right) \Bigg|_{r=0}^{2} \Bigg|_{\phi=0}^{2\pi} \Bigg|_{z=0}^{3} = 480\pi \text{ (mC)} = 1.5 \text{ C}.$$

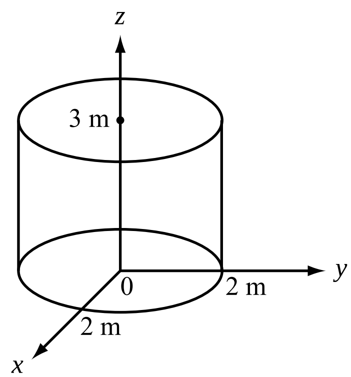

Figure P.2: Cylinder of Problem.2

Problem.3 Find the total charge contained in a cone def ned by $R \leq 2$ m and $0 \leq \theta \leq \pi/4$, given that $\rho_v = 10R^2 \cos^2\theta$ (mC/m^3).

Solution: For the cone of Fig. P.3,

$$Q = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \int_{R=0}^{2} 10R^2 \cos^2\theta \, R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= \left(\frac{-2}{3} R^5 \phi \cos^3\theta \right) \Bigg|_{R=0}^{2} \Bigg|_{\theta=0}^{\pi/4} \Bigg|_{\phi=0}^{2\pi}$$

$$= \frac{128\pi}{3} \left(1 - \left(\frac{\sqrt{2}}{2} \right)^3 \right) = 86.65 \text{ (mC)}.$$

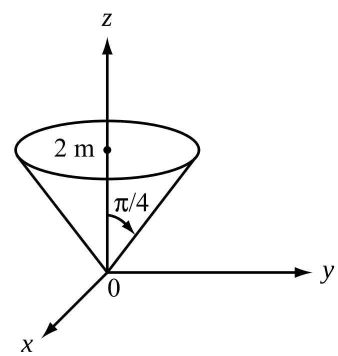

Figure P.3: Cone of Problem.3

Problem.4 If the line charge density is given by $\rho_l = 24y^2$ (mC/m), f nd the total charge distributed on the y-axis from $y = -5$ to $y = 5$.

Solution:

$$Q = \int_{-5}^{5} \rho_l \, dy = \int_{-5}^{5} 24y^2 \, dy = \left. \frac{24y^3}{3} \right|_{-5}^{5} = 2000 \text{ mC} = 2 \text{ C}.$$

Problem.5 Find the total charge on a circular disk def ned by $r \le a$ and $z = 0$ if:

(a) $\rho_s = \rho_{s0} \cos\phi$ (C/m^2),
(b) $\rho_s = \rho_{s0} \sin^2\phi$ (C/m^2),
(c) $\rho_s = \rho_{s0} e^{-r}$ (C/m^2),
(d) $\rho_s = \rho_{s0} e^{-r} \sin^2\phi$ (C/m^2),

where ρ_{s0} is a constant.

Solution:

(a)

$$Q = \int \rho_s \, ds = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \cos\phi \; r \, dr \, d\phi = \rho_{s0} \left. \frac{r^2}{2} \right|_0^a \left. \sin\phi \right|_0^{2\pi} = 0.$$

(b)

$$Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \sin^2\phi \; r \, dr \, d\phi = \rho_{s0} \left. \frac{r^2}{2} \right|_0^a \int_0^{2\pi} \left(\frac{1 - \cos 2\phi}{2} \right) d\phi$$

$$= \frac{\rho_{s0} a^2}{4} \left. \left(\phi - \frac{\sin 2\phi}{2} \right) \right|_0^{2\pi} = \frac{\pi a^2}{2} \rho_{s0}.$$

(c)

$$Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} r\, dr\, d\phi = 2\pi\rho_{s0} \int_0^a re^{-r}\, dr$$
$$= 2\pi\rho_{s0} \left[-re^{-r} - e^{-r}\right]_0^a$$
$$= 2\pi\rho_{s0}[1 - e^{-a}(1+a)].$$

(d)

$$Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} \sin^2\phi \; r\, dr\, d\phi$$
$$= \rho_{s0} \int_{r=0}^{a} re^{-r}\, dr \int_{\phi=0}^{2\pi} \sin^2\phi\, d\phi$$
$$= \rho_{s0}[1 - e^{-a}(1+a)] \cdot \pi = \pi\rho_{s0}[1 - e^{-a}(1+a)].$$

Problem.6 If $\mathbf{J} = \hat{\mathbf{y}}4xz$ (A/m^2), f nd the current I f owing through a square with corners at $(0,0,0)$, $(2,0,0)$, $(2,0,2)$, and $(0,0,2)$.

Solution: the net current f owing through the square shown in Fig.
P.6 is

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_{x=0}^{2} \int_{z=0}^{2} (\hat{\mathbf{y}}4xz)\Big|_{y=0} \cdot (\hat{\mathbf{y}}\, dx\, dz) = \left(x^2 z^2\right)\Big|_{x=0}^{2}\Big|_{z=0}^{2} = 16 \text{ A}.$$

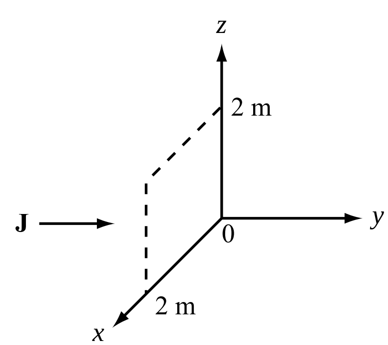

Figure P.6: Square surface.

Problem.7 If $\mathbf{J} = \hat{\mathbf{R}}5/R$ (A/m^2), f nd I through the surface $R = 5$ m.

Solution: Using Eq. (4.12), we have

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(\hat{\mathbf{R}} \frac{5}{R} \right) \cdot (\hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi)$$

$$= -5R\phi \cos\theta \Big|_{R=5} \Big|_{\theta=0}^{\pi} \Big|_{\phi=0}^{2\pi} = 100\pi = 314.2 \quad \text{(A)}.$$

Problem.8 An electron beam shaped like a circular cylinder of radius r_0 carries a charge density given by

$$\rho_v = \left(\frac{-\rho_0}{1+r^2} \right) \quad \text{(C/m}^3\text{)},$$

where ρ_0 is a positive constant and the beam's axis is coincident with the z-axis.

(a) Determine the total charge contained in length L of the beam.

(b) If the electrons are moving in the $+z$-direction with uniform speed u, determine the magnitude and direction of the current crossing the z-plane.

Solution:

(a)

$$Q = \int_{r=0}^{r_0} \int_{z=0}^{L} \rho_v \, d\mathcal{V} = \int_{r=0}^{r_0} \int_{z=0}^{L} \left(\frac{-\rho_0}{1+r^2} \right) 2\pi r \, dr \, dz$$

$$= -2\pi\rho_0 L \int_0^{r_0} \frac{r}{1+r^2} \, dr = -\pi\rho_0 L \ln(1+r_0^2).$$

(b)

$$\mathbf{J} = \rho_v \mathbf{u} = -\hat{\mathbf{z}} \frac{u\rho_0}{1+r^2} \quad \text{(A/m}^2\text{)},$$

$$I = \int \mathbf{J} \cdot d\mathbf{s}$$

$$= \int_{r=0}^{r_0} \int_{\phi=0}^{2\pi} \left(-\hat{\mathbf{z}} \frac{u\rho_0}{1+r^2} \right) \cdot \hat{\mathbf{z}} r \, dr \, d\phi$$

$$= -2\pi u\rho_0 \int_0^{r_0} \frac{r}{1+r^2} \, dr = -\pi u\rho_0 \ln(1+r_0^2) \quad \text{(A)}.$$

Current direction is along $-\hat{\mathbf{z}}$.

Section-2: Coulomb's Law

Problem.9 A square with sides 2 m each has a charge of 40 μC at each of its four corners. Determine the electric f eld at a point 5 m above the center of the square.

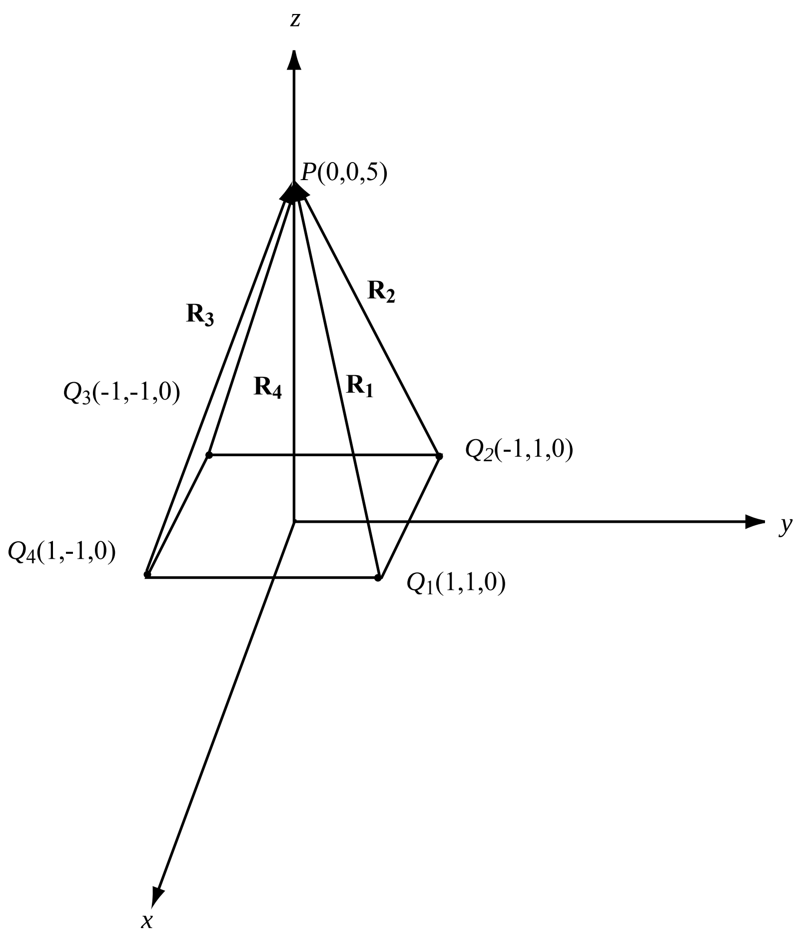

Figure P.9: Square with charges at the corners.

Solution: The distance $|R|$ between any of the charges and point P is

$|R| = \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27}.$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R}_1}{|\mathbf{R}|^3} + \frac{\mathbf{R}_2}{|\mathbf{R}|^3} + \frac{\mathbf{R}_3}{|\mathbf{R}|^3} + \frac{\mathbf{R}_4}{|\mathbf{R}|^3}\right]$$
$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}} + \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}} + \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}} + \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}5}{(27)^{3/2}}\right]$$
$$= \hat{\mathbf{z}}\frac{5Q}{(27)^{3/2}\pi\varepsilon_0} = \hat{\mathbf{z}}\frac{5 \times 40\ \mu\text{C}}{(27)^{3/2}\pi\varepsilon_0} = \frac{1.42}{\pi\varepsilon_0} \times 10^{-6}\ (\text{V/m}) = \hat{\mathbf{z}}51.2\ (\text{kV/m}).$$

Problem.10 Three point charges, each with $q =$ 3 nC, are located at the corners of a triangle in the x–y plane, with one corner at the origin, another at $(2\text{ cm}, 0, 0)$, and the third at $(0, 2\text{ cm}, 0)$. Find the force acting on the charge located at the origin.

Solution: the electric f eld at the origin due to the other two point charges [Fig. P.10]:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\left[\frac{3\text{ nC}\ (-\hat{\mathbf{x}}0.02)}{(0.02)^3}\right] + \frac{3\text{ nC}\ (-\hat{\mathbf{y}}0.02)}{(0.02)^3} = -67.4(\hat{\mathbf{x}} + \hat{\mathbf{y}})\ (\text{kV/m}) \text{ at } \mathbf{R} = 0.$$

the force $\mathbf{F} = q\mathbf{E} = -202.2(\hat{\mathbf{x}} + \hat{\mathbf{y}})\ (\mu\text{N}).$

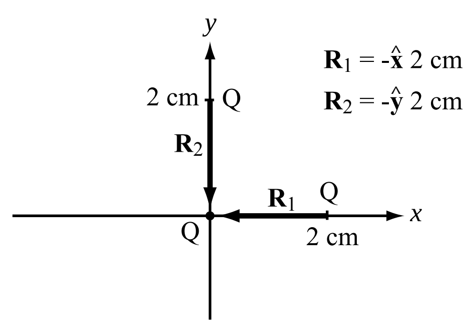

Figure P.10: Locations of charges in Problem.10.

Problem.11 Charge $q_1 =$ 6 μC is located at $(1\text{ cm}, 1\text{ cm}, 0)$ and charge q_2 is located at $(0, 0, 4\text{ cm})$. What should q_2 be so that $\mathbf{E}$ at $(0, 2\text{ cm}, 0)$ has no y-component?

Solution: For the conf guration of Fig. P.11,

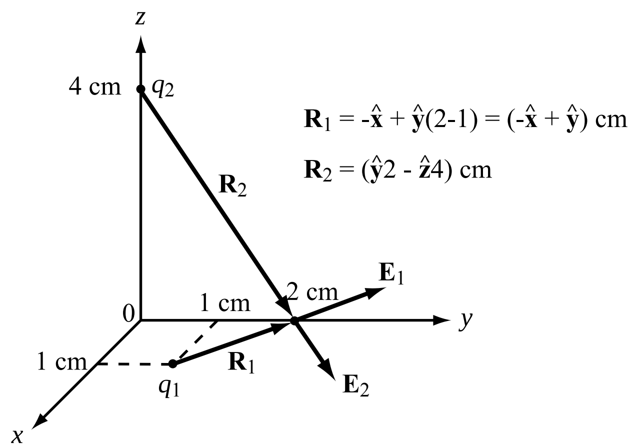

Figure P.11: Locations of charges in Problem.11.

$$\mathbf{E}(\mathbf{R}=\hat{\mathbf{y}}2\text{cm}) = \frac{1}{4\pi\varepsilon}\left[\frac{6\mu\text{C}(-\hat{\mathbf{x}}+\hat{\mathbf{y}})\times 10^{-2}}{(2\times 10^{-2})^{3/2}} + \frac{q_2(\hat{\mathbf{y}}2-\hat{\mathbf{z}}4)\times 10^{-2}}{(20\times 10^{-2})^{3/2}}\right]$$
$$= \frac{1}{4\pi\varepsilon}[-\hat{\mathbf{x}}21.21\times 10^{-6}+\hat{\mathbf{y}}(21.21\times 10^{-6}+0.224q_2)$$
$$-\hat{\mathbf{z}}0.447q_2] \quad \text{(V/m)}.$$

If $E_y = 0$, then $q_2 = -21.21\times 10^{-6}/0.224 \approx -94.69$ (μC).

Problem.12 A line of charge with uniform density $\rho_l =$ 8 (μC/m) exists in air along the z-axis between $z=0$ and $z=5$ cm. Find $\mathbf{E}$ at (0,10 cm,0).

Solution: for the line of charge shown in Fig. P.12 gives

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{l'}\hat{\mathbf{R}}'\frac{\rho_l\,dl'}{R'^2},$$
$$R' = \hat{\mathbf{y}}0.1-\hat{\mathbf{z}}z$$
$$= \frac{1}{4\pi\varepsilon_0}\int_{z=0}^{0.05}(8\times 10^{-6})\frac{(\hat{\mathbf{y}}0.1-\hat{\mathbf{z}}z)}{[(0.1)^2+z^2]^{3/2}}\,dz$$
$$= \frac{8\times 10^{-6}}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{y}}10z+\hat{\mathbf{z}}}{\sqrt{(0.1)^2+z^2}}\right]\Bigg|_{z=0}^{0.05}$$
$$= 71.86\times 10^3\,[\hat{\mathbf{y}}4.47-\hat{\mathbf{z}}1.06] = \hat{\mathbf{y}}321.4\times 10^3-\hat{\mathbf{z}}76.2\times 10^3 \quad \text{(V/m)}.$$

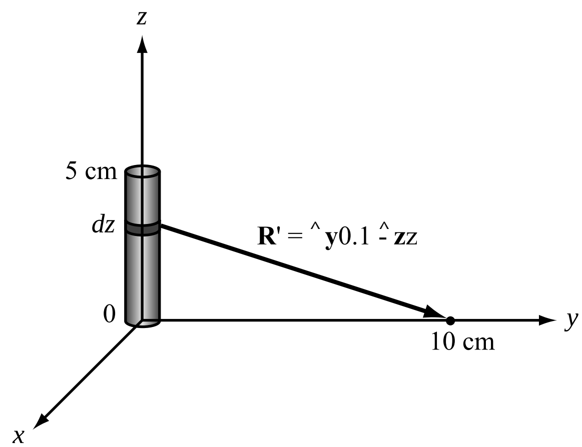

Figure P.12: Line charge.

Problem.13 Electric charge is distributed along an arc located in the x–y plane and defined by $r = 2$ cm and $0 \leq \phi \leq \pi/4$. If $\rho_l = 5$ (μC/m), find $\mathbf{E}$ at $(0,0,z)$ and then evaluate it at (a) the origin, (b) $z = 5$ cm, and (c) $z = -5$ cm.

Solution: For the arc of charge shown in Fig. P.13, $dl = r\,d\phi = 0.02\,d\phi$, and $\mathbf{R}' = -\hat{\mathbf{x}}0.02\cos\phi - \hat{\mathbf{y}}0.02\sin\phi + \hat{\mathbf{z}}z$.

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\varepsilon_0}\int_{l'} \hat{\mathbf{R}}' \frac{\rho_l\,dl'}{R'^2} \\
&= \frac{1}{4\pi\varepsilon_0}\int_{\phi=0}^{\pi/4} \rho_l \frac{(-\hat{\mathbf{x}}0.02\cos\phi - \hat{\mathbf{y}}0.02\sin\phi + \hat{\mathbf{z}}z)}{((0.02)^2+z^2)^{3/2}}\,0.02\,d\phi \\
&= \frac{898.8}{\left((0.02)^2+z^2\right)^{3/2}}\left[-\hat{\mathbf{x}}0.014 - \hat{\mathbf{y}}0.006 + \hat{\mathbf{z}}0.78z\right] \quad \text{(V/m)}.
\end{aligned}$$

(a) At $z = 0$, $\mathbf{E} = -\hat{\mathbf{x}}\,1.6 - \hat{\mathbf{y}}\,0.66$ (MV/m).
(b) At $z = 5$ cm, $\mathbf{E} = -\hat{\mathbf{x}}\,81.4 - \hat{\mathbf{y}}\,33.7 + \hat{\mathbf{z}}\,226$ (kV/m).
(c) At $z = -5$ cm, $\mathbf{E} = -\hat{\mathbf{x}}\,81.4 - \hat{\mathbf{y}}\,33.7 - \hat{\mathbf{z}}\,226$ (kV/m).

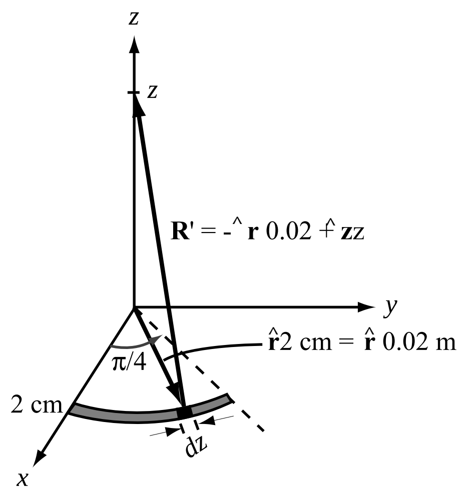

Figure P .13: Line charge along an arc.

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Section-3: Gauss's Law

Problem.14 Three inf nite lines of charge, all parallel to the z-axis, are located at the three corners of the kite-shaped arrangement shown in Fig.-14. If the two right triangles are symmetrical and of equal corresponding sides, show that the electric f eld is zero at the origin.

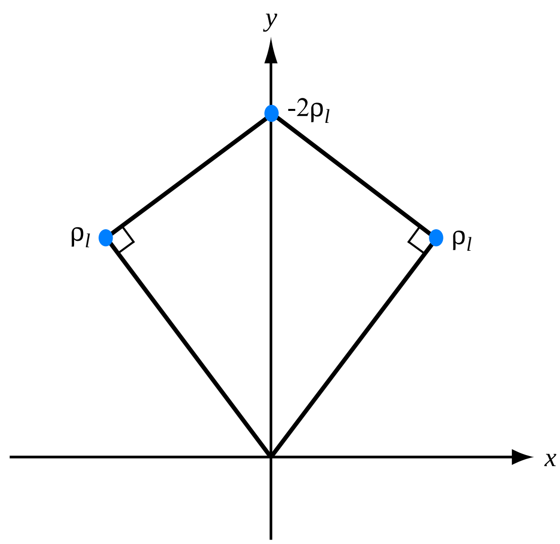


Figure P.14: Kite-shaped arrangment of line charges for Problem.14.

Solution: The f eld due to an inf nite line of charge is given by the present case, the total **E** at the origin is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3.$$

The components of $\mathbf{E}_1$ and $\mathbf{E}_2$ along $\hat{\mathbf{x}}$ cancel and their components along $-\hat{\mathbf{y}}$ add. Also, $\mathbf{E}_3$ is along $\hat{\mathbf{y}}$ because the line charge on the y-axis is negative. Hence,

$$\mathbf{E} = -\hat{\mathbf{y}}\,\frac{2\rho_l \cos\theta}{2\pi\varepsilon_0 R_1} + \hat{\mathbf{y}}\,\frac{2\rho_l}{2\pi\varepsilon_0 R_2}\,.$$

But $\cos\theta = R_1/R_2$. Hence,

$$\mathbf{E} = -\hat{\mathbf{y}}\,\frac{\rho_l}{\pi\varepsilon_0 R_1}\,\frac{R_1}{R_2} + \hat{\mathbf{y}}\,\frac{\rho_l}{\pi\varepsilon_0 R_2} = 0.$$

Problem.15 Three inf nite lines of charge, $\rho_{l_1} =$ 3 (nC/m), $\rho_{l_2} = -3$ (nC/m), and $\rho_{l_3} = 3$ (nC/m), are all parallel to the z-axis. If they pass through the respective points

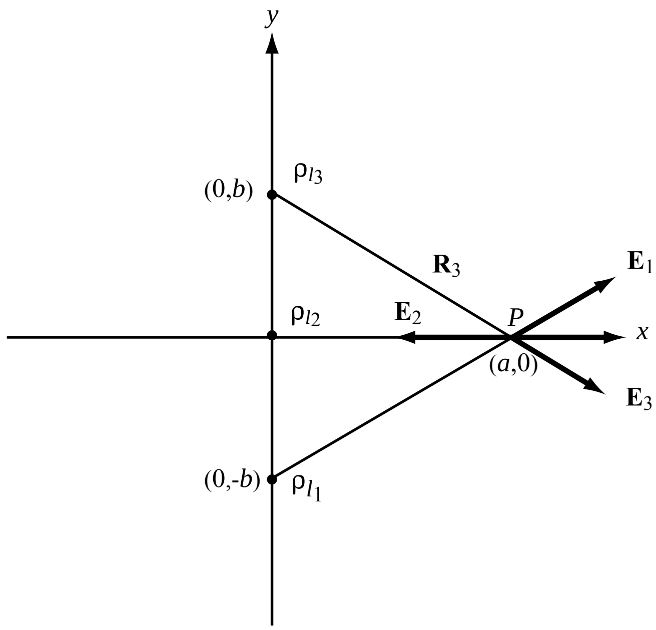


Figure P.15: Three parallel line charges.

$(0,-b)$, $(0,0)$, and $(0,b)$ in the x–y plane, f nd the electric f eld at $(a,0,0)$. Evaluate your result for $a = 2$ cm and $b = 1$ cm.

Solution:

$$\begin{aligned}
\rho_{l_1} &= 3 \quad \text{(nC/m)}, \\
\rho_{l_2} &= -3 \quad \text{(nC/m)}, \\
\rho_{l_3} &= \rho_{l_1}, \\
\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3.
\end{aligned}$$

Components of line charges 1 and 3 along y cancel and components along x add. Hence,

$$\mathbf{E} = \hat{\mathbf{x}}\,\frac{2\rho_{l_1}}{2\pi\varepsilon_0 R_1}\cos\theta + \hat{\mathbf{x}}\,\frac{\rho_{l_2}}{2\pi\varepsilon_0 a}\,.$$

with $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$ and $R_1 = \sqrt{a^2+b^2}$,

$$\mathbf{E} = \frac{\hat{\mathbf{x}}\,3}{2\pi\varepsilon_0}\left[\frac{2a}{a^2+b^2} - \frac{1}{a}\right] \times 10^{-9} \quad \text{(V/m)}.$$

For $a = 2$ cm and $b = 1$ cm,

$$\mathbf{E} = \hat{\mathbf{x}}\, 1.62 \quad \text{(kV/m)}.$$

Problem.16 A horizontal strip lying in the x–y plane is of width d in the y-direction and infinitely long in the x-direction. If the strip is in air and has a uniform charge distribution ρ_s, use Coulomb's law to obtain an explicit expression for the electric field at a point P located at a distance h above the centerline of the strip. Extend your result to the special case where d is infinite.

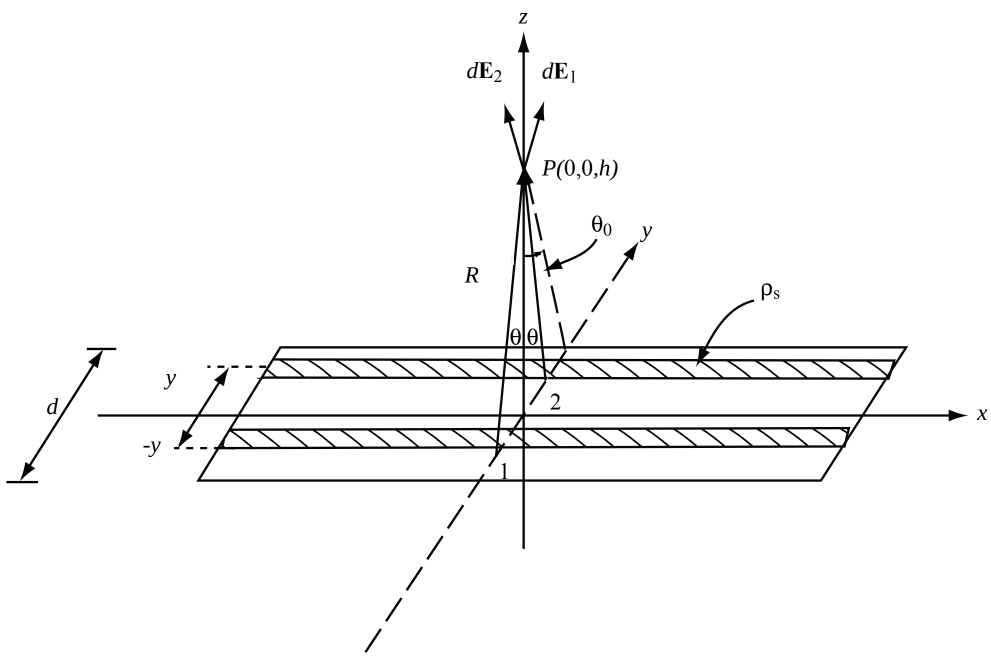


Figure P.16: Horizontal strip of charge.

Solution: The strip of charge density ρ_s (C/m^2) can be treated as a set of adjacent line charges each of charge $\rho_l = \rho_s\, dy$ and width dy. At point P, the fields of line charge at distance y and line charge at distance $-y$ give contributions that cancel each other along $\hat{\mathbf{y}}$ and add along $\hat{\mathbf{z}}$. For each such pair,

$$d\mathbf{E} = \hat{\mathbf{z}}\, \frac{2\rho_s\, dy \cos\theta}{2\pi\varepsilon_0 R}.$$

With $R = h/\cos\theta$, we integrate from $y = 0$ to $d/2$, which corresponds to $\theta = 0$ to $\theta_0 = \sin^{-1}[(d/2)/(h^2 + (d/2)^2)^{1/2}]$. Thus,

$$\mathbf{E} = \int_0^{d/2} d\mathbf{E} = \hat{\mathbf{z}}\frac{\rho_s}{\pi\varepsilon_0}\int_0^{d/2}\frac{\cos\theta}{R}\,dy = \hat{\mathbf{z}}\frac{\rho_s}{\pi\varepsilon_0}\int_0^{\theta_0}\frac{\cos^2\theta}{h}\cdot\frac{h}{\cos^2\theta}\,d\theta$$
$$= \hat{\mathbf{z}}\frac{\rho_s}{\pi\varepsilon_0}\theta_0.$$

For an inf nitely wide sheet, $\theta_0 = \pi/2$ and $\mathbf{E} = \hat{\mathbf{z}}\dfrac{\rho_s}{2\varepsilon_0}$,

.

Problem.17 Given the electric f ux density

$$\mathbf{D} = \hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y) \quad (\text{C/m}^2),$$

determine

(a) ρ_v,

(b) the total charge Q enclosed in a cube 2 m on a side, located in the f rst octant with three of its sides coincident with the x-, y-, and z-axes and one of its corners at the origin, and

(c) the total charge Q in the cube,

Solution:

(a) By applying

$$\rho_v = \nabla\cdot\mathbf{D} = \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(3x-2y) = 0.$$

(b) Integrate the charge density over the volume:

$$Q = \int_{\mathcal{V}}\nabla\cdot\mathbf{D}\,d\mathcal{V} = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} 0\,dx\,dy\,dz = 0.$$

(c) Apply Gauss' law to calculate the total charge

$$Q = \oint\mathbf{D}\cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}},$$

$$F_{\text{front}} = \int_{y=0}^{2}\int_{z=0}^{2}\left(\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)\right)\Big|_{x=2}\cdot(\hat{\mathbf{x}}\,dz\,dy)$$
$$= \int_{y=0}^{2}\int_{z=0}^{2} 2(x+y)\Big|_{x=2}\,dz\,dy = \left(2z\left(2y+\frac{1}{2}y^2\right)\Big|_{z=0}^{2}\right)\Big|_{y=0}^{2} = 24,$$

$$F_{\text{back}} = \int_{y=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\Big|_{x=0} \cdot (-\hat{\mathbf{x}}\,dz\,dy)$$

$$= -\int_{y=0}^{2}\int_{z=0}^{2} 2(x+y)\Big|_{x=0} dz\,dy = -\left(zy^2\Big|_{z=0}^{2}\right)\Big|_{y=0}^{2} = -8,$$

$$F_{\text{right}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\Big|_{y=2} \cdot (\hat{\mathbf{y}}\,dz\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\Big|_{y=2} dz\,dx = \left(z\left(\frac{3}{2}x^2 - 4x\right)\Big|_{z=0}^{2}\right)\Big|_{x=0}^{2} = -4,$$

$$F_{\text{left}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\Big|_{y=0} \cdot (-\hat{\mathbf{y}}\,dz\,dx)$$

$$= -\int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\Big|_{y=0} dz\,dx = -\left(z\left(\frac{3}{2}x^2\right)\Big|_{z=0}^{2}\right)\Big|_{x=0}^{2} = -12,$$

$$F_{\text{top}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\Big|_{z=2} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\Big|_{z=2} dy\,dx = 0,$$

$$F_{\text{bottom}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\Big|_{z=0} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\Big|_{z=0} dy\,dx = 0.$$

Thus $Q = \oint \mathbf{D} \cdot d\mathbf{s} = 24 - 8 - 4 - 12 + 0 + 0 = 0.$

Problem.18 Repeat Problem.17 for $\mathbf{D} = \hat{\mathbf{x}}xy^3z^3$ (C/m^2).

Solution:

(a) $\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(xy^3z^3) = y^3z^3.$

(b) Total charge Q is given by:

$$Q = \int_{\mathcal{V}} \nabla \cdot \mathbf{D}\,d\mathcal{V} = \int_{z=0}^{2}\int_{y=0}^{2}\int_{x=0}^{2} y^3z^3\,dx\,dy\,dz = \frac{xy^4z^4}{16}\Big|_{x=0}^{2}\Big|_{y=0}^{2}\Big|_{z=0}^{2} = 32 \text{ C}.$$

(c) Using Gauss' law we have

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}.$$

Note that $\mathbf{D} = \hat{\mathbf{x}} D_x$, so only F_{front} and F_{back} (integration over $\hat{\mathbf{z}}$ surfaces) will contribute to the integral.

$$\begin{aligned}
F_{\text{front}} &= \int_{z=0}^{2} \int_{y=0}^{2} (\hat{\mathbf{x}} x y^3 z^3)\bigg|_{x=2} \cdot (\hat{\mathbf{x}}\, dy\, dz) \\
&= \int_{z=0}^{2} \int_{y=0}^{2} x y^3 z^3 \bigg|_{x=2} dy\, dz = \left(2 \left(\frac{y^4 z^4}{16} \right) \bigg|_{y=0}^{2} \right) \bigg|_{z=0}^{2} = 32\,, \\
F_{\text{back}} &= \int_{z=0}^{2} \int_{y=0}^{2} (\hat{\mathbf{x}} x y^3 z^3)\bigg|_{x=0} \cdot (-\hat{\mathbf{x}}\, dy\, dz) = -\int_{z=0}^{2} \int_{y=0}^{2} x y^3 z^3 \bigg|_{x=0} dy\, dz = 0.
\end{aligned}$$

Thus $Q = \oint \mathbf{D} \cdot d\mathbf{s} = 32 + 0 + 0 + 0 + 0 + 0 = 32$ C.

Problem.19 Charge Q_1 is uniformly distributed over a thin spherical shell of radius a, and charge Q_2 is uniformly distributed over a second spherical shell of radius b, with $b > a$. Apply Gauss's law to f nd $\mathbf{E}$ in the regions $R < a$, $a < R < b$, and $R > b$.

Solution: Using symmetry considerations, we know $\mathbf{D} = \hat{\mathbf{R}} D_R$. From, $d\mathbf{s} = \hat{\mathbf{R}} R^2 \sin\theta\, d\theta\, d\phi$ for an element of a spherical surface. Using Gauss's law in integral form,

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{\text{tot}},$$

where Q_{tot} is the total charge enclosed in S. For a spherical surface of radius R,

$$\begin{aligned}
\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (\hat{\mathbf{R}} D_R) \cdot (\hat{\mathbf{R}} R^2 \sin\theta\, d\theta\, d\phi) &= Q_{\text{tot}}, \\
D_R R^2 (2\pi) [-\cos\theta]_0^{\pi} &= Q_{\text{tot}}, \\
D_R &= \frac{Q_{\text{tot}}}{4\pi R^2}\,.
\end{aligned}$$

we know a linear, isotropic material has the constitutive relationship $\mathbf{D} = \varepsilon \mathbf{E}$. Thus, we f nd $\mathbf{E}$ from $\mathbf{D}$.

(a) In the region $R < a$,

$$Q_{\text{tot}} = 0, \qquad \mathbf{E} = \hat{\mathbf{R}}E_{\text{R}} = \frac{\hat{\mathbf{R}}Q_{\text{tot}}}{4\pi R^2\varepsilon} = 0 \quad \text{(V/m)}.$$

(b) In the region $a < R < b$,

$$Q_{\text{tot}} = Q_1, \qquad \mathbf{E} = \hat{\mathbf{R}}E_{\text{R}} = \frac{\hat{\mathbf{R}}Q_1}{4\pi R^2\varepsilon} \quad \text{(V/m)}.$$

(c) In the region $R > b$,

$$Q_{\text{tot}} = Q_1 + Q_2, \qquad \mathbf{E} = \hat{\mathbf{R}}E_{\text{R}} = \frac{\hat{\mathbf{R}}(Q_1 + Q_2)}{4\pi R^2\varepsilon} \quad \text{(V/m)}.$$

Problem.20 The electric f ux density inside a dielectric sphere of radius a centered at the origin is given by

$$\mathbf{D} = \hat{\mathbf{R}}\rho_0 R \quad \text{(C/m}^2\text{)},$$

where ρ_0 is a constant. Find the total charge inside the sphere.

Solution:

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \hat{\mathbf{R}}\rho_0 R \cdot \hat{\mathbf{R}}R^2 \sin\theta \, d\theta \, d\phi \bigg|_{R=a}$$
$$= 2\pi\rho_0 a^3 \int_0^{\pi} \sin\theta \, d\theta = -2\pi\rho_0 a^3 \cos\theta\big|_0^{\pi} = 4\pi\rho_0 a^3 \quad \text{(C)}.$$

Problem.21 In a certain region of space, the charge density is given in cylindrical coordinates by the function:

$$\rho_{\text{v}} = 50re^{-r} \quad \text{(C/m}^3\text{)}.$$

Apply Gauss's law to f nd **D**.

Solution:

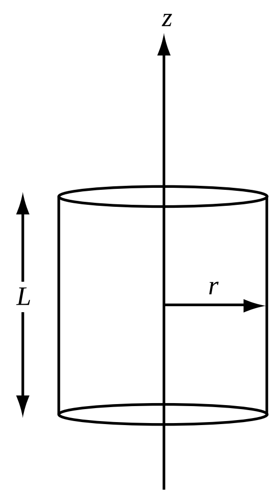


Figure P.21: Gaussian surface.

Method 1: Integral Form of Gauss's Law

Since ρ_v varies as a function of r only, so will $\mathbf{D}$. Hence, we construct a cylinder of radius r and length L, coincident with the z-axis. Symmetry suggests that $\mathbf{D}$ has the functional form $\mathbf{D} = \hat{\mathbf{r}} D$. Hence,

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q,$$

$$\int \hat{\mathbf{r}} D \cdot d\mathbf{s} = D(2\pi r L),$$

$$Q = 2\pi L \int_0^r 50 r e^{-r} \cdot r\, dr$$

$$= 100\pi L[-r^2 e^{-r} + 2(1 - e^{-r}(1+r))],$$

$$\mathbf{D} = \hat{\mathbf{r}} D = \hat{\mathbf{r}}\, 50 \left[\frac{2}{r}(1 - e^{-r}(1+r)) - r e^{-r} \right].$$

Method 2: Differential Method

$$\nabla \cdot \mathbf{D} = \rho_v, \qquad \mathbf{D} = \hat{\mathbf{r}} D_r,$$

with D_r being a function of r.

$$\frac{1}{r} \frac{\partial}{\partial r}(r D_r) = 50 r e^{-r},$$

$$\frac{\partial}{\partial r}(rD_r) = 50r^2 e^{-r},$$

$$\int_0^r \frac{\partial}{\partial r}(rD_r)\,dr = \int_0^r 50r^2 e^{-r}\,dr,$$

$$rD_r = 50[2(1 - e^{-r}(1+r)) - r^2 e^{-r}],$$

$$\mathbf{D} = \hat{\mathbf{r}}\,rD_r = \hat{\mathbf{r}}\,50\left[\frac{2}{r}(1 - e^{-r}(1+r)) - re^{-r}\right].$$

Problem.22 An infinitely long cylindrical shell extending between $r = 1$ m and $r = 3$ m contains a uniform charge density ρ_{v0}. Apply Gauss's law to find **D** in all regions.

Solution: For $r < 1$ m, $\mathbf{D} = 0$.

For $1 \le r \le 3$ m,

$$\oint_S \hat{\mathbf{r}} D_r \cdot d\mathbf{s} = Q,$$

$$D_r \cdot 2\pi r L = \rho_{v0} \cdot \pi L(r^2 - 1^2),$$

$$\mathbf{D} = \hat{\mathbf{r}} D_r = \hat{\mathbf{r}}\,\frac{\rho_{v0}\pi L(r^2-1)}{2\pi r L} = \hat{\mathbf{r}}\,\frac{\rho_{v0}(r^2-1)}{2r}, \qquad 1 \le r \le 3 \text{ m}.$$

For $r \ge 3$ m,

$$D_r \cdot 2\pi r L = \rho_{v0}\pi L(3^2 - 1^2) = 8\rho_{v0}\pi L,$$

$$\mathbf{D} = \hat{\mathbf{r}} D_r = \hat{\mathbf{r}}\,\frac{4\rho_{v0}}{r}, \qquad r \ge 3 \text{ m}.$$

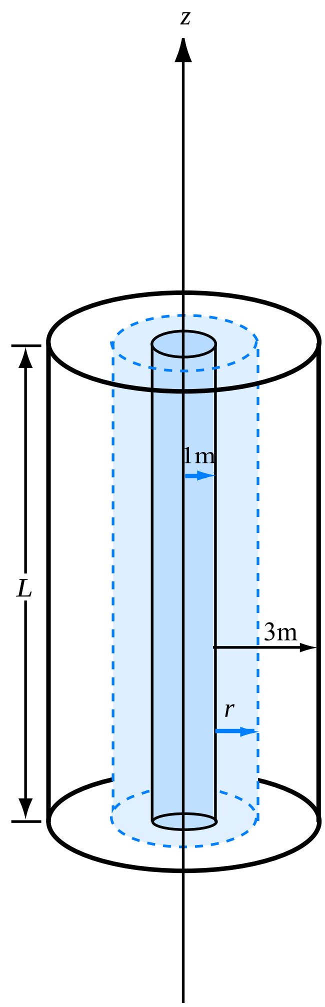


Figure P.22: Cylindrical shell.

Problem.23 If the charge density increases linearly with distance from the origin such that $\rho_v = 0$ at the origin and $\rho_v = 40$ C/m^3 at $R = 2$ m, f nd the corresponding variation of $\mathbf{D}$.

Solution:

$$\rho_v(R) = a + bR,$$
$$\rho_v(0) = a = 0,$$

$$\rho_v(2) = 2b = 40.$$

Hence, $b = 20$.

$$\rho_v(R) = 20R \quad (\text{C/m}^3).$$

Applying Gauss's law to a spherical surface of radius R,

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho_v \, d\mathcal{V},$$

$$D_R \cdot 4\pi R^2 = \int_0^R 20R \cdot 4\pi R^2 \, dR = 80\pi \frac{R^4}{4},$$

$$D_R = 5R^2 \quad (\text{C/m}^2),$$

$$\mathbf{D} = \hat{\mathbf{R}} D_R = \hat{\mathbf{R}} 5R^2 \quad (\text{C/m}^2).$$

Section-4: Electric Potential

Problem.24 A square in the x–y plane in free space has a point charge of $+Q$ at corner $(a/2, a/2)$ and the same at corner $(a/2, -a/2)$ and a point charge of $-Q$ at each of the other two corners.

(a) Find the electric potential at any point P along the x-axis.

(b) Evaluate V at $x = a/2$.

Solution: $R_1 = R_2$ and $R_3 = R_4$.

$$V = \frac{Q}{4\pi\varepsilon_0 R_1} + \frac{Q}{4\pi\varepsilon_0 R_2} + \frac{-Q}{4\pi\varepsilon_0 R_3} + \frac{-Q}{4\pi\varepsilon_0 R_4} = \frac{Q}{2\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_3}\right)$$

with

$$R_1 = \sqrt{\left(x - \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2},$$

$$R_3 = \sqrt{\left(x + \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}.$$

At $x = a/2$,

$$R_1 = \frac{a}{2},$$

$$R_3 = \frac{a\sqrt{5}}{2},$$

$$V = \frac{Q}{2\pi\varepsilon_0}\left(\frac{2}{a} - \frac{2}{\sqrt{5}\,a}\right) = \frac{0.55Q}{\pi\varepsilon_0 a}.$$

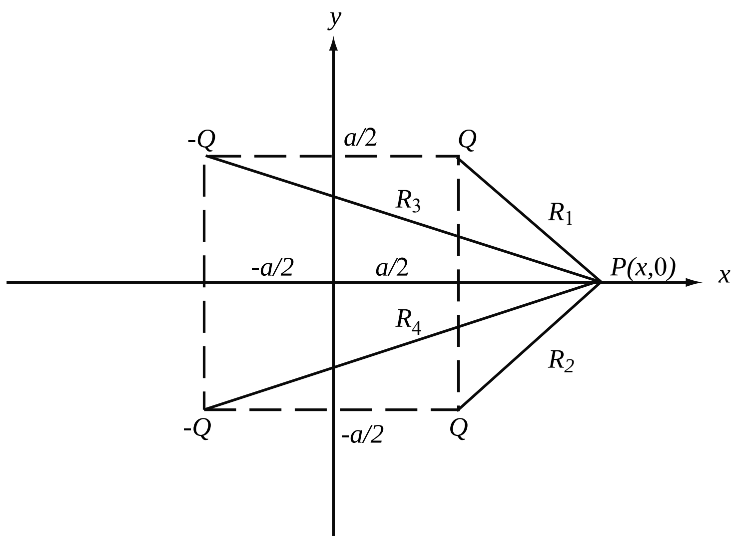


Figure P.24: Potential due to four point charges.

Problem.25 The circular disk of radius a shown in Fig. (P.25) has uniform charge density ρ_s across its surface.

(a) Obtain an expression for the electric potential V at a point $P(0,0,z)$ on the z-axis.

(b) Use your result to f nd **E** and then evaluate it for $z = h$. Compare your f nal expression with result which was obtained on the basis of Coulomb's law.

Solution:

(a) Consider a ring of charge at a radial distance r. The charge contained in width dr is

$$dq = \rho_s(2\pi r\,dr) = 2\pi\rho_s r\,dr.$$

The potential at P is

$$dV = \frac{dq}{4\pi\varepsilon_0 R} = \frac{2\pi\rho_s r\,dr}{4\pi\varepsilon_0 (r^2+z^2)^{1/2}}.$$

The potential due to the entire disk is

$$V = \int_0^a dV = \frac{\rho_s}{2\varepsilon_0}\int_0^a \frac{r\,dr}{(r^2+z^2)^{1/2}} = \frac{\rho_s}{2\varepsilon_0}(r^2+z^2)^{1/2}\bigg|_0^a = \frac{\rho_s}{2\varepsilon_0}\left[(a^2+z^2)^{1/2}-z\right].$$

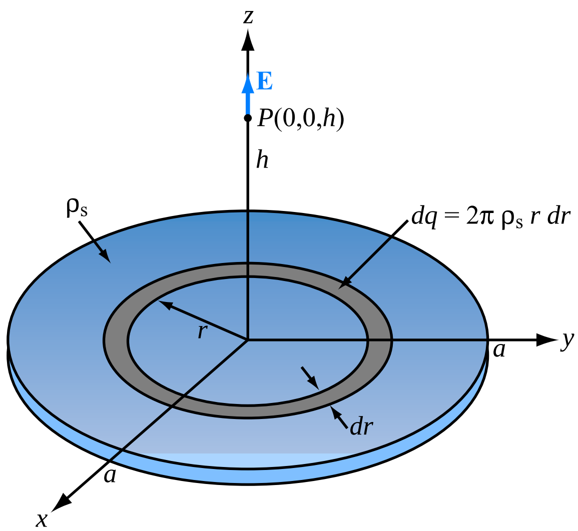


Figure P.25: Circular disk of charge.

(b)

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{x}}\,\frac{\partial V}{\partial x} - \hat{\mathbf{y}}\,\frac{\partial V}{\partial y} - \hat{\mathbf{z}}\,\frac{\partial V}{\partial z} = \hat{\mathbf{z}}\,\frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right].$$

The expression for **E** reduces to??????????? when $z = h$.

√

√

√

Problem.26 For the electric dipole shown in Fig. 4-13, $d = 1$ cm and $|\mathbf{E}| = 4$ (mV/m) at $R = 1$ m and $\theta = 0^\circ$. Find $\mathbf{E}$ at $R = 2$ m and $\theta = 90^\circ$.

Solution: For $R = 1$ m and $\theta = 0^\circ$, $|\mathbf{E}| = 4$ mV/m, we can solve for q using:

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}(\hat{\mathbf{R}} 2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta).$$

Hence,

$$|\mathbf{E}| = \left(\frac{qd}{4\pi\varepsilon_0}\right) 2 = 4 \text{ mV/m} \quad \text{at } \theta = 0^\circ,$$

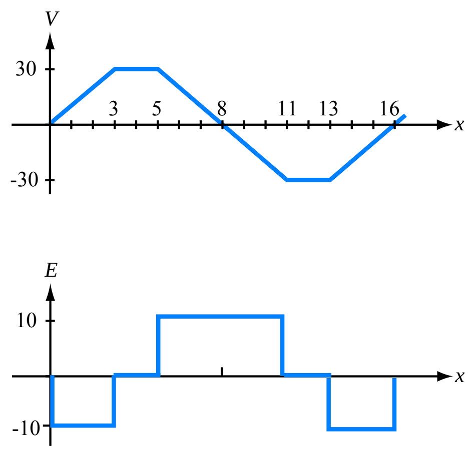
$$q = \frac{10^{-3} \times 8\pi\varepsilon_0}{d} = \frac{10^{-3} \times 8\pi\varepsilon_0}{10^{-2}} = 0.8\pi\varepsilon_0 \quad \text{(C)}.$$

Again to f nd $\mathbf{E}$ at $R = 2$ m and $\theta = 90^\circ$, we have

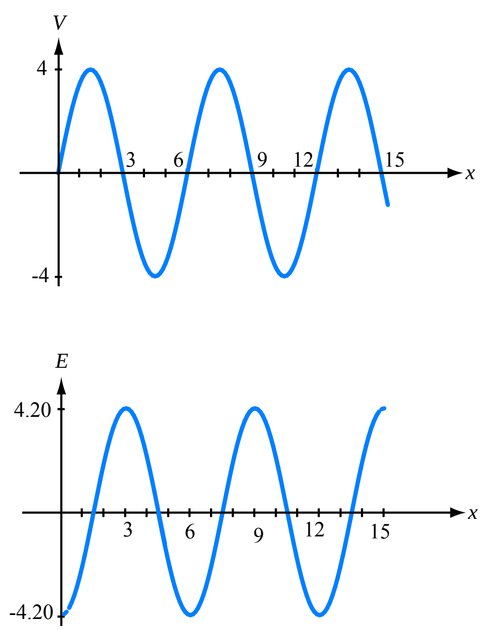
$$\mathbf{E} = \frac{0.8\pi\varepsilon_0 \times 10^{-2}}{4\pi\varepsilon_0 \times 2^3}(\hat{\mathbf{R}}(0) + \hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{\theta}}\frac{1}{4} \quad \text{(mV/m)}.$$

Problem.27 For each of the following distributions of the electric potential V, sketch the corresponding distribution of **E** (in all cases, the vertical axis is in volts and the horizontal axis is in meters):

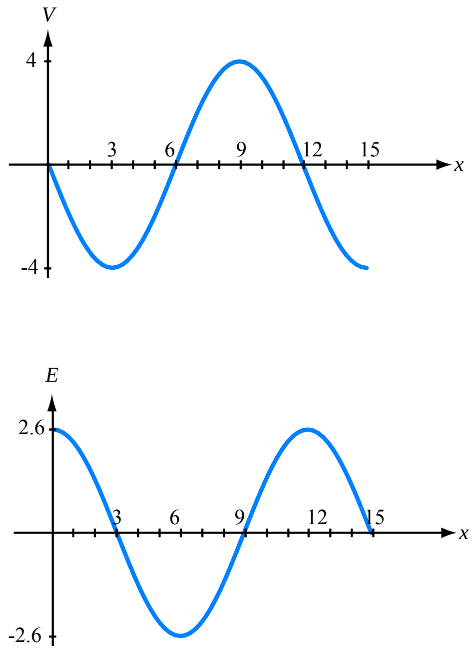
Solution:



(a)



(b)



(c)

Figure P.27: Electric potential distributions

Problem.28 Given the electric f eld

$$\mathbf{E} = \hat{\mathbf{R}}\,\frac{18}{R^2} \quad \text{(V/m)},$$

f nd the electric potential of point A with respect to point B where A is at $+2$ m and B at -4 m, both on the z-axis.

Solution:

$$V_{AB} = V_A - V_B = -\int_B^A \mathbf{E}\cdot d\mathbf{l}.$$

Along z-direction, $\hat{\mathbf{R}} = \hat{\mathbf{z}}$ and $\mathbf{E} = \hat{\mathbf{z}}\,\frac{18}{z^2}$ for $z \geq 0$, and $\hat{\mathbf{R}} = -\hat{\mathbf{z}}$ and $\mathbf{E} = -\hat{\mathbf{z}}\,\frac{18}{z^2}$ for $z \leq 0$. Hence,

$$V_{AB} = -\int_{-4}^{2} \hat{\mathbf{R}}\frac{18}{z^2}\cdot\hat{\mathbf{z}}\,dz = -\left[\int_{-4}^{0} -\hat{\mathbf{z}}\,\frac{18}{z^2}\cdot\hat{\mathbf{z}}\,dz + \int_{0}^{2} \hat{\mathbf{z}}\,\frac{18}{z^2}\cdot\hat{\mathbf{z}}\,dz\right] = 4 \text{ V}.$$

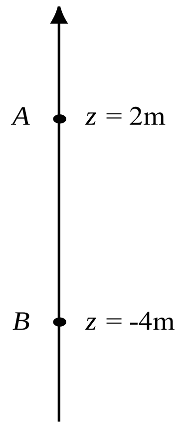


Figure P.28: Potential between B and A.

Problem.29 An infnitely long line of charge with uniform density $\rho_l =$ 9 (nC/m) lies in the x–y plane parallel to the y-axis at $x = 2$ m. Find the potential V_{AB} at point $A(3\text{ m}, 0, 4\text{ m})$ in Cartesian coordinates with respect to point $B(0,0,0)$
.

Solution: According to,

$$V = \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{r_2}{r_1}\right)$$

where r_1 and r_2 are the distances of A and B. In this case,

$$r_1 = \sqrt{(3-2)^2 + 4^2} = \sqrt{17}\text{ m},$$
$$r_2 = 2\text{ m}.$$

Hence,

$$V_{AB} = \frac{9 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}} \ln\left(\frac{2}{\sqrt{17}}\right) = -117.09\text{ V}.$$

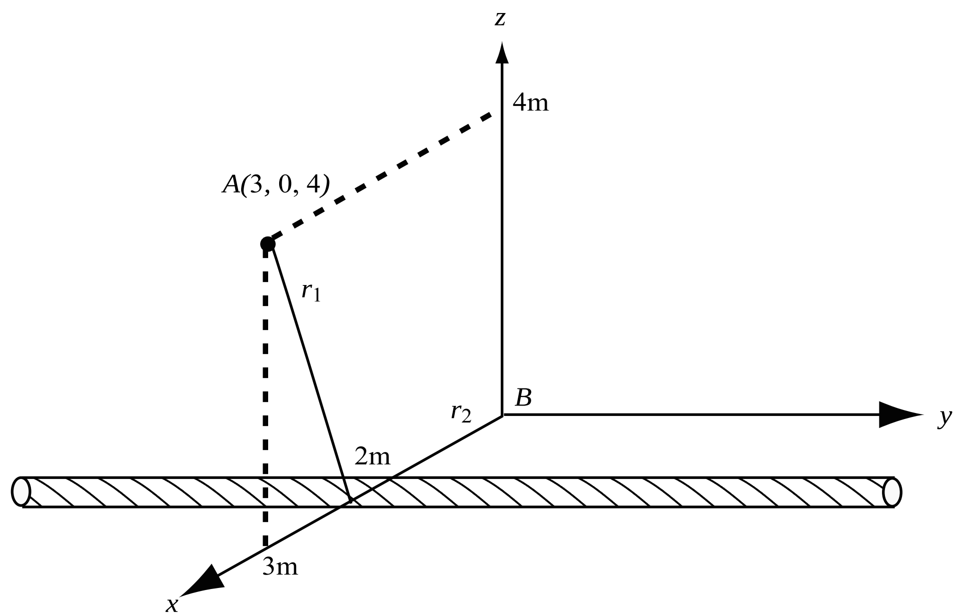


Figure P.29: Line of charge parallel to y-axis.

Problem.30 The x–y plane contains a uniform sheet of charge with $\rho_{s_1} = 0.2$ (nC/m^2) and a second sheet with $\rho_{s_2} = -0.2$ (nC/m^2) occupies the plane $z = 6$ m. Find V_{AB}, V_{BC}, and V_{AC} for $A(0,0,6\text{ m})$, $B(0,0,0)$, and $C(0,-2\text{ m},2\text{ m})$.

Solution: We start by f nding the **E** f eld in the region between the plates. For any point above the x–y plane, $\mathbf{E}_1$ due to the charge on x–y plane is,,

$$\mathbf{E}_1 = \hat{\mathbf{z}}\frac{\rho_{s_1}}{2\varepsilon_0}.$$

In the region below the top plate, **E** would point downwards for positive ρ_{s_2} on the top plate. In this case, $\rho_{s_2} = -\rho_{s_1}$. Hence,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \hat{\mathbf{z}}\frac{\rho_{s_1}}{2\varepsilon_0} - \hat{\mathbf{z}}\frac{\rho_{s_2}}{2\varepsilon_0} = \hat{\mathbf{z}}\frac{2\rho_{s_1}}{2\varepsilon_0} = \hat{\mathbf{z}}\frac{\rho_{s_1}}{\varepsilon_0}.$$

Since **E** is along $\hat{\mathbf{z}}$, only change in position along z can result in change in voltage.

$$V_{AB} = -\int_0^6 \hat{\mathbf{z}}\frac{\rho_{s_1}}{\varepsilon_0}\cdot\hat{\mathbf{z}}\,dz = -\frac{\rho_{s_1}}{\varepsilon_0}z\bigg|_0^6 = -\frac{6\rho_{s_1}}{\varepsilon_0} = -\frac{6\times 0.2\times 10^{-9}}{8.85\times 10^{-12}} = -135.59\text{ V}.$$

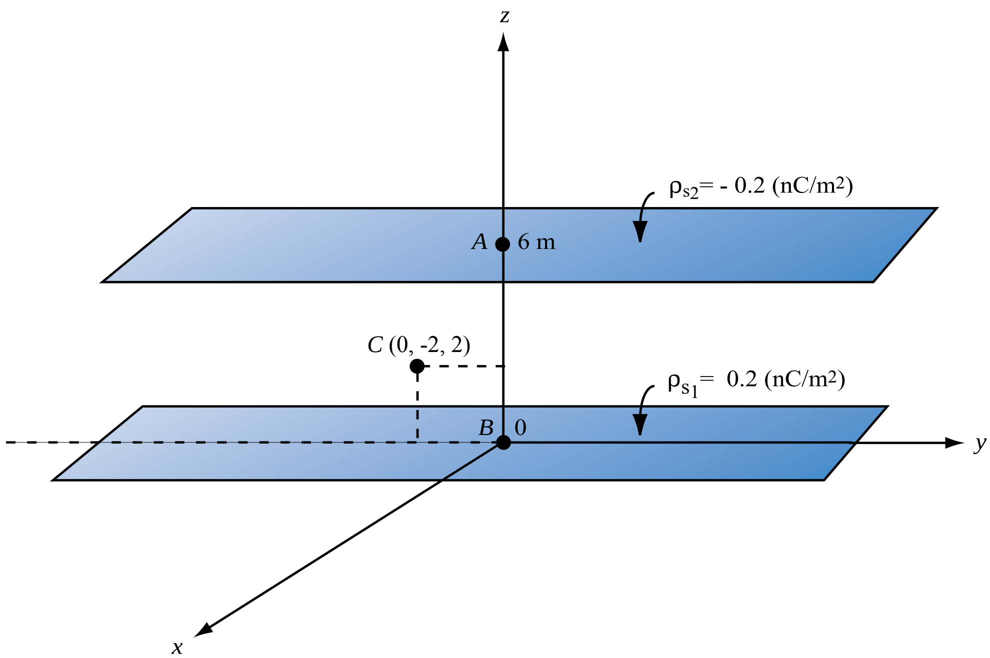


Figure P.30: Two parallel planes of charge.

The voltage at C depends only on the z-coordinate of C. Hence, with point A being at the lowest potential and B at the highest potential,

$$V_{BC} = \frac{-2}{6} V_{AB} = -\frac{(-135.59)}{3} = 45.20 \text{ V},$$

$$V_{AC} = V_{AB} + V_{BC} = -135.59 + 45.20 = -90.39 \text{ V}.$$